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Algebra I, Worksheet 2

Exercise n°1 : Let the set $A = \{1, 2, 3\}$. Determine whether the following assertions are True or False.

$$3 \in A, \quad 3 \subset A, \quad \phi \in A, \quad \{\{1,2\}, 3\} = A, \quad \{1,2\} \subset A, \quad A \cup \left\{\phi\right\} = A$$

Exercise n°2 : Let A and B be two sets.
1. Prove the following properties of power sets :
(a). if $A \subset B$, then $\mathcal{P}(A) \subset \mathcal{P}(B)$.
(b). $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
(c). $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$.
2. Give an example of two sets A and B such that $\mathcal{P}(A \cup B) \not\subset \mathcal{P}(A) \cup \mathcal{P}(B)$.

Exercise n°3 : Let A, B, C be subsets of a set E. Prove

$$1. A \subset B \Longrightarrow C_E^B \subset C_E^A \qquad 2.\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

$$3.\ A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C) \qquad 4.\ A \triangle B = (A \cup B) \backslash (A \cap B)$$

Then determine $A \triangle A, A \triangle \phi, A \triangle E$.

Exercise n°4 : Determine whether each of the following relations $\mathfrak{R}$ is reflexive, symmetric, antisymmetric, or transitive.
1. $\forall x, y \in \mathbb{R} : x \mathfrak{R}_1 y \Longleftrightarrow \cos^2 x + \sin^2 y = 1$.
2. $\forall x, y \in \mathbb{R} : x \mathfrak{R}_2 y \Longleftrightarrow |x| = \left|y\right|$.

Exercise n°5 : Let $\mathfrak{R}$ be the binary relation on $\mathbb{Z}$ defined by

$$\forall x, y \in \mathbb{Z} : x \mathfrak{R} y \Longleftrightarrow \exists k \in \mathbb{Z} : x + 2y = 3k.$$

1. Prove that $\mathfrak{R}$ is an equivalence relation on the set $\mathbb{Z}$.
2. For $x \in \mathbb{Z}$, determine the equivalence class of x, denoted by $\dot{x}$.
3. Determine the quotient set $\mathbb{Z}/\mathfrak{R}$.

Exercise n°6 : We define a relation $\mathfrak{R}$ on $\mathbb{N}^*$ as follows

$$\forall x, y \in \mathbb{N}^* : x \mathfrak{R} y \Longleftrightarrow \exists n \in \mathbb{N}^* : y = x^n.$$

This relation can also be expressed as "y is a non-zero integer power of x".
1. Prove that $\mathfrak{R}$ is a partial order relation on $\mathbb{N}^*$.
2. Let $A = \{2, 4, 16\}$ be a subset of $\mathbb{N}^*$. Examine the existence of a greatest element and a least element in A (denoted $\max(A)$ and $\min(A)$) with respect to the relation $\mathfrak{R}$.

Exercise n°7 : (Supplementary Exercise). Let E be a set, and let $A \subset E$. Define a binary relation $\mathfrak{R}$ on $\mathcal{P}(E)$ (the power set of E) by

$$\forall X, Y \in \mathcal{P}(E) : X \mathfrak{R} Y \Longleftrightarrow X \cap A = Y \cap A.$$

1. Prove that $\mathfrak{R}$ is an equivalence relation on $\mathcal{P}(E)$.
2. For a subset X of E denote by $\dot{X}$ the equivalence class of X under $\mathfrak{R}$. Describe $\dot{X}$ explicitly, and determine the equivalence classes of the sets ϕ, A, E, and C_E^A.

Exercise n°8 : (Supplementary Exercise). In $\mathbb{R}^2$, consider the binary relation $\mathfrak{R}$ defined by

$$\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 : (x_1, y_1) \mathfrak{R} (x_2, y_2) \iff x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

1. Prove that $\mathfrak{R}$ is an order relation. Is this order total ?
2. Let $A = \{(1, 2), (3, 1)\} \subset \mathbb{R}^2$. Determine the set of lower bounds $\mathcal{LB}(A, E)$, the set of upper bounds $\mathcal{UB}(A, E)$, as well as the infimum $\inf(A)$ and supremum $\sup(A)$ of A.