CHAPTER 2: IMPROPER INTEGRALS

Definition of an Improper Integral:

The integral $\int_a^b f(x)\,dx$ is called an improper integral if:

- $a = -\infty$ or $b = +\infty$ or both.
- We can find multiple points within the same interval as long as $a \le x \le b$. These points are called the singular points of $f(x)$.

Improper Integral of the First Kind:

Let $f(x)$ be a bounded and integrable function over every finite interval $a \le x \le b$. Then, by definition:

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx$$

The integral $\int_a^{+\infty} f(x)\,dx$ is said to be convergent if $\lim_{b\to+\infty} \int_a^b f(x)\,dx$ exists.

The integral $\int_a^{+\infty} f(x)\,dx$ is said to be divergent if $\lim_{b\to+\infty} \int_a^b f(x)\,dx$ does not exist.

The same definition applies for:

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

Improper Integral of the First Kind for Specific Functions:

- The geometric or exponential integral: $\int_a^{+\infty} e^{-tx}\,dx$, where t is a constant, is convergent if $t > 0$, and divergent if $t \le 0$.
- The power integral of the first kind: $\int_a^{+\infty} \frac{dx}{x^p}$ where p is a constant and $a > 0$, is convergent if $p > 1$, and divergent if $p \le 1$. (This is known as the Riemann series.)
- The power integral of the first kind: $\int_0^{1} \frac{dx}{x^p}$ where p is a constant and $a = 0$ is a singularity point, is convergent if $p < 1$, and divergent if $p \ge 1$. (This is known as the Riemann series.)

Convergence Criteria for Improper Integrals of the First Kind:

- Comparison Criterion for Integrals with Non-Negative Integrands:

a) Convergence: Let $g(x) \geq 0$ for all $x \geq a$, and suppose that $\int_a^{+\infty} g(x)\,dx$ converges. Then, if $0 \leq f(x) \leq g(x)$ for all $x \geq a$, $\int_a^{+\infty} f(x)\,dx$ also converges.

b) Divergence: Let $g(x) \geq 0$ for all $x \geq a$, and suppose that $\int_a^{+\infty} g(x)\,dx$ diverges. Then, if $f(x) \geq g(x)$ for all $x \geq a$, $\int_a^{+\infty} f(x)\,dx$ also diverges.

- Quotient Criterion for Integrals with Non-negative Integrands:

a) If $f(x) \geq 0$ and $g(x) \geq 0$, and if $\lim\limits_{x\to+\infty} \frac{f(x)}{g(x)} = A \neq 0$ or ∞, then $\int_a^{+\infty} f(x)\,dx$ and $\int_a^{+\infty} g(x)\,dx$ either both converge or both diverge.

b) If $A = 0$ in (a) and if $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converges.

c) If $A = \infty$ in (a) and if $\int_a^{+\infty} g(x)\,dx$ diverges, then $\int_a^{+\infty} f(x)\,dx$ diverges.

This criterion is related to the comparison criterion, of which it is a very useful alternative form. In particular, by taking $g(x) = \frac{1}{x^p}$, we obtain, based on the known behavior of this integral:

Theorem 1: let $\lim\limits_{x\to+\infty} x^p f(x) = A$, then:

- $\int_a^{+\infty} f(x)\,dx$ converges if $p > 1$ and A est fini.
- $\int_a^{+\infty} f(x)\,dx$ diverges if $p \leq 1$ and $A \neq 0$ (A may be infinite).

- Absolute Convergence and semi-convergence: $\int_a^{+\infty} f(x)\,dx$ is said to be absolutely convergent if $\int_a^{+\infty} |f(x)|\,dx$ converges. If $\int_a^{+\infty} f(x)\,dx$ converges but $\int_a^{+\infty} |f(x)|\,dx$ diverges, then $\int_a^{+\infty} f(x)\,dx$ is said to be semi-convergent.