

Chapter 1: Simple Integrals and Multiple Integrals

Multiple Integrals and Area Calculation

Double Integral:

Let $z = F(x, y)$ be a function defined in a closed and bounded domain D of the xoy plane. We decompose the domain D arbitrarily into n elementary domains of surfaces $\Delta_{s_1}, \Delta_{s_2}, \dots, \Delta_{s_n}$. Now, we choose an arbitrary point $(P_k)_{1 \leq k \leq n}$ in each elementary domain.

Let $F(P_1), F(P_2), \dots, F(P_n)$ be the values of the function $F(x, y)$ at these points, and then form the products $F(P_k)\Delta_{s_k}$. The sum of these products is called the integral sum of the function $F(x, y)$ over the domain D , and it takes the following form:

$$\sum_{k=1}^n F(P_k)\Delta_{s_k} = F(P_1)\Delta_{s_1} + F(P_2)\Delta_{s_2} + \dots + F(P_n)\Delta_{s_n}$$

The double integral of the function $F(x, y)$ over the domain D is defined as the limit of the integral sum when the largest of the domains $\Delta_{s_k} \rightarrow 0$. It is denoted as:

$$\iint F(x, y) dx dy = \iint F(P) ds = \lim_{\max \Delta_{s_k} \rightarrow 0} \sum_{k=1}^n F(P_k) \Delta_{s_k}$$

Rule for calculating a double integral:

Let $F(x, y)$ be a function defined and continuous in a closed domain D of $\mathbb{R}^2$, and this domain is such that:

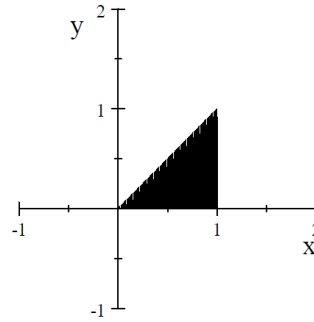
$$\begin{cases} \text{si } a \leq x \leq b & \Rightarrow f_1(x) \leq y \leq f_2(x) \\ & \text{ou} \\ \text{si } c \leq y \leq d & \Rightarrow g_1(y) \leq x \leq g_2(y) \end{cases}$$

Then, the double integral of the function F over D is calculated as follows:

$$\iint F(x, y) dx dy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} F(x, y) dy \right] dx = \int_c^d \left[\int_{g_1(y)}^{g_2(y)} F(x, y) dx \right] dy$$

Example: Calculate the following double integral: $I_1 = \iint \frac{x}{\sqrt{y}} dx dy$

Where the domain D is defined by: $D = \{(x, y) \in \mathbb{R}^2 / 0 < y \leq x < 1\}$



Let $(x, y) \in D$, we choose $0 < x < 1$, and it follows that $0 < y \leq x$. Thus,

$$I_1 = \int_0^1 \left[\int_0^x \frac{x}{\sqrt{y}} dy \right] dx = \int_0^1 x [2\sqrt{y}]_0^x dx = 2 \int_0^1 (x\sqrt{x}) dx = 2 \left(\frac{2}{5} x^{5/2} \right)_0^1 = \frac{4}{5}$$

Fubini's Theorem or Integration over a Rectangle:

Let $F(x, y)$ be a function defined and continuous in a rectangle $D = [a, b] \times [c, d]$ of $\mathbb{R}^2$:

Then,

$$\iint F(x, y) dx dy = \int_a^b \left[\int_c^d F(x, y) dy \right] dx = \int_c^d \left[\int_a^b F(x, y) dx \right] dy$$

Example: Calculate the following double integral: $I_2 = \iint (x^2 + y) dx dy$

If the domain $D = [0, 1] \times [1, 2]$:

According to Fubini's Theorem, we have:

$$\begin{aligned} \iint (x^2 + y) dx dy &= \int_0^1 \left[\int_1^2 (x^2 + y) dy \right] dx = \int_0^1 \left[x^2 y + \frac{1}{2} y^2 \right]_1^2 dx = \int_0^1 \left(x^2 + \frac{3}{2} \right) dx \\ &= \left(\frac{1}{3} x^3 + \frac{3}{2} x \right)_0^1 = \frac{11}{6} \end{aligned}$$

Change of Variables in a Double Integral

Double Integral in Curvilinear Coordinates:

Suppose the integration variables x and y can be expressed as functions of new variables u and v as follows:

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$$

Where the functions $x(u, v)$ and $y(u, v)$ have continuous partial derivatives in a domain D' of the $u'o'v$ plane, and the Jacobian of the transformation in the domain D' does not cancel:

$$J = \left| \frac{D(x, y)}{D(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

Then, the formula for transforming a double integral to curvilinear coordinates is as follows:

$$\iint F(x, y) dx dy = \iint F(x(u, v), y(u, v)) |J| du dv$$

Example: Perform the change of variable indicated in the following integral:

$$I_4 = \int_0^1 dx \int_x^{2x} F(x, y) dy, \quad \begin{cases} u = x + y \\ v = \frac{y}{x + y} \end{cases}$$

The new values of x and y in terms of (u, v) are:

$$\begin{cases} x = u - uv \\ y = uv \end{cases}$$

So,

$$J = \begin{vmatrix} 1 - v & -u \\ v & u \end{vmatrix} = u$$

The new domain D' is:

$$D' = \{(u, v) \in \mathbb{R}^2, \frac{1}{2} \leq v \leq \frac{2}{3}, \quad 0 \leq u \leq \frac{1}{1-v}\}$$

Then the integral I_4 is:

$$I_4 = \int_{\frac{1}{2}}^{\frac{2}{3}} dv \int_0^{\frac{1}{1-v}} u G(u, v) du$$

Double Integral in Polar Coordinates

When transitioning from rectangular (Cartesian) coordinates x and y to polar coordinates r and θ , we define:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \text{ avec } r > 0 \text{ et } 0 \leq \theta \leq 2\pi$$

$$\text{On a : } J = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

The double integral of a function F in polar coordinates is:

$$\iint F(x, y) dx dy = \iint F(r \cos \theta, r \sin \theta) r dr d\theta$$

Example : Calculate : $I_5 = \iint \frac{dx dy}{x^2 + y^2}$ where the domain D is defined by :

$$D = \{(x, y) \in \mathbb{R}^2, 4 \leq x^2 + y^2 \leq 9\}$$

We have the polar coordinates as: $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$

$$4 \leq x^2 + y^2 \leq 9, 2 \leq r \leq 3 \text{ de plus } 0 \leq \theta \leq 2\pi$$

We have :

$$I_5 = \iint \frac{dx dy}{x^2 + y^2} = \iint \frac{r dr d\theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = \int_2^3 \frac{1}{r} dr \int_0^{2\pi} d\theta = 2\pi [\ln r]_2^3 = \pi \ln \frac{9}{4}$$

Area Calculation in $\mathbb{R}^2$

The area of a plane figure bounded by the domain D is calculated using the formula $A(D)$:

$$A(D) = \iint dx dy$$

Area Calculation in $\mathbb{R}^3$

Let S be a uniform surface given by the equation $z = f(x; y)$; then the area of the surface S is expressed by the formula:

$$A(S) = \iint \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy$$