Set Theory

Set theory is a branch of mathematical logic concerned with the study of sets, which are collections of abstract mathematical objects and the operations applied to them. Interest in this field began in the 19th century with Georg Cantor. Since then, many branches of mathematics have used set theory to develop the field and introduce new concepts. Cantor's formulation of the mathematical set concept was somewhat limited, yet the set theory concept as he formulated it remains valid for most uses. The modern concept of a mathematical set differs from Cantor's by providing precise axioms to avoid the limitations of the original definition. The modern definition does not contradict Cantor's but clarifies it to prevent paradoxes like Russell's paradox or the Barber paradox.

0.1 Russell's Paradox

Russell's paradox or the Barber paradox are two names for the same idea, attributed to the British philosopher Bertrand Russell in set theory. People have been preoccupied with resolving this paradox because set theory is foundational to mathematics; if a contradiction exists, a vast amount of mathematical knowledge would lose validity.

The Barber paradox considers the following scenario: "In a village, there is a barber who shaves only those people who do not shave themselves." The question is: Does the barber shave himself? Any attempted answer leads to a contradiction.

If the barber shaves himself, the statement says he shaves only those who do not shave themselves ? a contradiction. If he does not shave himself, then he must be shaved by the barber (himself) ? again a contradiction.

Now, let's discuss Russell's paradox, the mathematical formulation of the Barber paradox. Consider a set E containing sets that satisfy the condition: "Every set that does not contain itself is an element of E."

Applying this condition to sets within E poses no issue. However, applying it to E itself leads to:
1. If $E \in E$, this contradicts the condition "every set that does not contain itself is an element of E."
2. If $E \notin E$, then E satisfies the condition "every set that does not contain itself is an element of E," implying $E \in E$? a contradiction.

This reveals a flaw in set theory. Russell himself proposed a theory (theory of types) to resolve this paradox: when discussing classes of elements, we specify conditions for elements, not for the set of elements.

Summary

To avoid ambiguity in set theory, a precise definition of a set via axioms is established:
1. A set is uniquely determined if membership is well-defined, i.e., we can unambiguously determine the truth of $a \in E$ or $a \notin E$.
2. A mathematical object cannot be both a set and an element simultaneously (e.g., $a \in a$ is invalid).
3. The "set of all sets" does not exist.

0.2 Concepts and Operations on Sets

Subset

For two sets E and F:

$$E \subset F \equiv (\forall x)(x \in E \Rightarrow x \in F)$$

Equality

For two sets E and F:

$$E = F \equiv (\forall x)(x \in E \Leftrightarrow x \in F)$$

Empty Set

The set containing no elements, denoted by $\emptyset$.

Results

1. $E \subset E$
2. $\emptyset \subset E$
3. $E = F \Leftrightarrow (E \subset F) \wedge (F \subset E)$
4. The empty set is unique.
5. $(E \subset F) \wedge (F \subset G) \Rightarrow (E \subset G)$ (transitivity of inclusion).

Definition

Let E and F be arbitrary sets.

1. **Intersection:**

$$E \cap F = \{x \mid x \in E \wedge x \in F\}$$

2. **Union:**

$$E \cup F = \{x \mid x \in E \vee x \in F\}$$

3. **Set Difference:**

$$E - F = \{x \mid x \in E \wedge x \notin F\}$$

4. **Symmetric Difference:**

$$E \triangle F = (E \cup F) - (E \cap F)$$

Properties

For arbitrary sets A, B, C:

1. $A \cap \emptyset = \emptyset$; $A \cup \emptyset = A$; $A \cap A = A$; $A \cup A = A$.
2. Commutativity: $A \cap B = B \cap A$; $A \cup B = B \cup A$.
3. Associativity: (i) $(A \cap B) \cap C = A \cap (B \cap C)$ (ii) $(A \cup B) \cup C = A \cup (B \cup C)$
4. Distributivity: (i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

0.3 Subsets, Power Set, Complement

Let E and A be sets.

Definition

A is a subset of E if $A \subset E$.

Definition

The power set of E, denoted $\mathcal{P}(E)$, is the set whose elements are all subsets of E:

$$A \in \mathcal{P}(E) \Leftrightarrow A \subset E$$

$$\mathcal{P}(E) = \{A \mid A \subset E\}$$

Note

1. $\emptyset \in \mathcal{P}(E)$
2. $E \in \mathcal{P}(E)$
3. If $\text{Card}(E) = n$, then $\text{Card}(\mathcal{P}(E)) = 2^n$.

Definition

Let A be a subset of E. The complement of A relative to E is:

$$C_E^A = \{x \mid x \in E \wedge x \notin A\} = E - A$$

Properties

1. $C_E^\emptyset = E$, $C_E^E = \emptyset$
2. $C_E(C_E^A) = A$
3. $C_E^{A\cap B} = C_E^A \cup C_E^B$; $C_E^{A\cup B} = C_E^A \cap C_E^B$

0.4 Cartesian Product of Two Sets

Let E and F be sets.

Definition

The Cartesian product of E and F is:

$$E \times F = \{(a, b) \mid a \in E, b \in F\}$$

The element (a, b) is called an ordered pair, and:

$$(a, b) = (a', b') \Leftrightarrow (a = a') \wedge (b = b')$$

Properties

1. $E \times \emptyset = \emptyset$
2. If $E \neq F$, then $E \times F \neq F \times E$.
3. If $\text{Card}(E) = m$, $\text{Card}(F) = n$, then $\text{Card}(E \times F) = m \times n$.

0.5 Equivalence Relations, Partitions

Let E be an arbitrary set and $\{A_i\}_{i\in I}$ (where I is an index set) a family of subsets of E.

Definition

A family $\{A_i\}_{i\in I}$ forms a partition of E if:

1. $E = \bigcup_{i\in I} A_i$
2. $A_i \cap A_j = \emptyset$ for all $i \neq j$, $(i, j) \in I$.

Example

Let $E = \{1, 2, 3\}$. 1. The family $\{\{1\}, \{2\}, \{3\}\}$ forms a partition of E. 2. The family $\{\{1\}, \{2, 3\}\}$ forms another partition of E.

Definition

Let E be an arbitrary set and $\{B_i\}_{i\in I}$ a family of arbitrary sets. The family $\{B_i\}_{i\in I}$ forms a cover of E if:

$$E \subset \bigcup_{i\in I} B_i$$

Example

1. Let $E = \{1, 2, 3\}$. The family $\{\{1\}, \{2, -1\}, \{3, 4, 5\}\}$ forms a cover of E. Similarly, $\{\{12\}, \{4, -1\}, \{3, 4, 7\}\}$ forms a cover of E. 2. Let $E = \mathbb{R}$. The family $\{B_n =]-n, n[, n \in \mathbb{N}\}$ forms a cover of E. The family $\{A_n =]n, n+1], n \in \mathbb{Z}\}$ forms both a partition and a cover simultaneously.

Definition

Let E be a set. A relation $\mathcal{R}$ on E is an equivalence relation if: 1. Reflexive: $\forall x \in E, x\mathcal{R}x$ 2. Symmetric: $\forall x, y \in E, x\mathcal{R}y \Rightarrow y\mathcal{R}x$ 3. Transitive: $\forall x, y, z \in E, x\mathcal{R}y \wedge y\mathcal{R}z \Rightarrow x\mathcal{R}z$

Definition

Let E be a set and $\mathcal{R}$ an equivalence relation on E. For each $x \in E$, the equivalence class of x is:

$$\bar{x} = \{y \in E \mid x\mathcal{R}y\}$$

The set of equivalence classes is:

$$\frac{E}{\mathcal{R}} = \{A \in \mathcal{P}(E) : \exists x \in E, A = \bar{x}\}$$

Theorem

Let E be a non-empty set and $\mathcal{R}$ an equivalence relation on E. The set of equivalence classes $\frac{E}{\mathcal{R}}$ forms a partition of E.

Theorem

Let E be a non-empty set and $\mathcal{P}$ a partition of E. The relation $\mathcal{R}$ defined by:

$$x\mathcal{R}y \Leftrightarrow \exists A \in \mathcal{P}(x \in A \wedge y \in A)$$

is an equivalence relation on E, and $\mathcal{P} = \frac{E}{\mathcal{R}}$.

0.6 Functions and Mappings

Definition

Let E and F be sets.
- A function f from E to F is a relation satisfying:
If $(x, y) \in f$ and $(x', y) \in f$, then $x = x'$.
- If $(x, y) \in f$:
$*$ x is a preimage of y. $*$ y is the image of x. Denoted by:

$$f : E \to F, \quad x \mapsto y = f(x)$$

- The domain of f is the set of elements with an image:

$$D(f) = \{x \in E : \exists y \in F, (x, y) \in f\}$$

- A mapping from E to F is a function with domain E.

Definition

Let E, F be sets. A function $f : E \to F$ is:

- Injective (one-to-one) if: $\forall x, y \in E, f(x) = f(y) \Rightarrow x = y$.

- Surjective (onto) if: $\forall y \in F, \exists x \in E, f(x) = y$.

- Bijective if both injective and surjective.

Definition

Let $f : E \to F$ be a function, $A \subset E$, $B \subset F$. - The direct image of A is:

$$f(A) = \{y \in F : \exists x \in A, f(x) = y\}$$

- The inverse image of B is:

$$f^{-1}(B) = \{x \in E : \exists y \in B, f(x) = y\}$$

Theorem

Let $f : E \to F$, $A_i \subset E$, $B_i \subset F$ for $i \in I$. Then:

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
2. $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$
3. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
4. $f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$
5. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$
6. $f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)$
7. $f^{-1}(C_F B) = C_E f^{-1}(B)$

0.7 Finite and Infinite Sets

Many sets allow (theoretically) counting their elements, though this may be impractical (e.g., counting water molecules on Earth). Conversely, some sets are infinite (e.g., natural numbers). For finite sets, we compare cardinalities; for infinite sets, we use bijections.

Definition : Countable Sets

A set is countable if there exists a bijection between it and the set of natural numbers.

Example :

1. The set of integers $\mathbb{Z}$ is countable (bijection to $\mathbb{N}$):

$$\mathbb{Z} \to \mathbb{N}, \quad n \mapsto \begin{cases} 2|n| & \text{if } n < 0 \\ 2n+1 & \text{if } n \geq 0 \end{cases}$$

2. The set of powers of 2 is countable: $\mathbb{N} \to \{2^n\}, n \mapsto 2^n$.

Properties :

1. Every subset of a countable set is finite or countable.
2. A finite or countable union of countable sets is countable.
3. Every infinite set contains a countable subset.

Definition : Equipotent Sets

Two sets M and N are equipotent (denoted $M \sim N$) if there exists a bijection between them.

Note :

1. Equipotence applies to finite and infinite sets.
2. A set is countable if equipotent to $\mathbb{N}$.
3. An infinite set can be equipotent to a proper subset (e.g., $\mathbb{N} \sim \mathbb{Z}$).

Example :

1. $\mathbb{N}$ and $\mathbb{Z}$ are equipotent.
2. The set of points in $[0,1]$ is equipotent to the real line.

Theorem : Cantor-Bernstein

Let A, B be sets with subsets $A_1 \subset A$, $B_1 \subset B$. If there exists a bijection $f : A \to B_1$ and a bijection $g : B \to A_1$, then A and B are equipotent.

0.8 Cardinality of a Set

1. For finite sets, cardinality is the number of elements.
2. The cardinality of a countable infinite set is denoted $\aleph_0$.
3. Sets equipotent to the real interval $[0,1]$ have the cardinality of the continuum, denoted $\mathfrak{c}$.

Summary :

Let A, B be sets with cardinalities $m(A)$, $m(B)$. To compare $m(A)$ and $m(B)$:

1. If A is equipotent to a subset of B, and B is equipotent to a subset of A, then $m(A) = m(B)$ (Cantor-Bernstein).

2. If A contains a subset equipotent to B, but B contains no subset equipotent to A, then $m(A) > m(B)$.

3. If B contains a subset equipotent to A, but A contains no subset equipotent to B, then $m(A) < m(B)$.

4. If neither contains a subset equipotent to the other, their cardinalities are incomparable. (This is forbidden by Zermelo's theorem).

0.9 Axiom of Choice and Zermelo's Theorem

The comparability of cardinalities of well-ordered sets raises the question: Can every set be well-ordered? Zermelo answered "yes," proving that any set can be well-ordered. This led to debates in mathematical logic and foundations, confirming the consistency and independence of Zermelo's Axiom. Zermelo's proof relies on the Axiom of Choice:

Definition : Axiom of Choice

Let A be an index set, and for each $i \in A$, let M_i be a non-empty set. Then there exists a function:

$$\phi : A \to \bigcup_{i \in A} M_i$$

such that $\phi(i) \in M_i$ for all $i \in A$ (i.e., ϕ "chooses" an element from each M_i).

Definition: Chain

Let M be an ordered set, $A \subset M$. A is a chain if every two elements in A are comparable.

Definition : Upper Bound

Let M be an ordered set, $A' \subset M$, $a \in M$. a is an upper bound of A' if every element of A' precedes a (i.e., $x \preccurlyeq a$ for all $\in A'$).

Theorem: Hausdorff

Every chain in a partially ordered set is contained in a maximal chain.

Lemma: Zorn

If every chain in a partially ordered set M has an upper bound, then every element of M precedes some maximal element.

Proofs of Zermelo's Theorem, Axiom of Choice, Hausdorff's Theorem, and Zorn's Lemma can be found in [14]. Rejecting Zermelo's theorem (i.e., claiming some sets cannot be well-ordered) undermines the Axiom of Choice and significantly weakens set theory. However, critiques led to important theories like recursion theory and computability.

0.10 Exercises

Exercise 1:

Let A, B be subsets of E. Prove:

1. $C_E(A \cup B) = C_E A \cap C_E B$
2. $C_E(A \cap B) = C_E A \cup C_E B$

Exercise 2:

Let A, B, C be subsets of E. Prove:

1. $A \triangle B = A \cap B \Leftrightarrow A = B = \emptyset$
2. $(A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cup (B \cap C) \cup (C \cap A)$
3. $A \triangle B = B \triangle A$
4. $A \triangle B = \emptyset \Leftrightarrow A = B$
5. $A \triangle (B \triangle C) = (A \triangle B) \triangle C$
6. $(A \triangle C = B \triangle C) \Leftrightarrow A = B$
7. $(A \cap B = A \cup B) \Rightarrow A = B$
8. $(A \cap B = A \cap C \wedge A \cup B = A \cup C) \Rightarrow B = C$

Exercise 3:

Let $f : E \to F$, $A_i \subset E$, $B_i \subset F$ for $i \in I$. Prove:

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
2. $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$
3. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
4. $f^{-1}\left(\bigcup_{i\in I} B_{i\,i}\right) = \bigcup_{i\in I} f^{-1}(B_i)$
5. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$
6. $f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)$
7. $f^{-1}(C_F B) = C_E f^{-1}(B)$