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Algebra I, Worksheet 3

Exercise n°1 : Let E and F be two non-empty sets, and let $\mathcal{R}$ be a binary relation from E to F. Determine which of the following relations define functions from E to F.
1. $E = \{0,1,2,5,6\}$, $F = \{1,2,3\}$, and $\Gamma_{\mathcal{R}_1} = \{(2,1),(6,3)\}$.
2. $E = \{1,2,3,4\}$, $F = \{3,5,6\}$, and $\Gamma_{\mathcal{R}_2} = \{(1,3),(1,5),(2,5)\}$.
3. $E = \{1,2,3,4\}$, $F = \{a,b,c\}$, and $\Gamma_{\mathcal{R}_2} = \{(1,c),(2,b),(3,a),(4,b)\}$.

Exercise n°2 : Let $f : E \times E \longrightarrow \mathbb{R}$ be an application such that

$$\forall a,b,c \in E : f(a,b) + f(b,c) + f(c,a) = 0.$$

Prove that the relation $\mathfrak{R}$ defined on E by

$$\forall a,b \in E : a\mathfrak{R}b \Longleftrightarrow f(a,b) = 0,$$

is an equivalence relation.

Exercise n°3 : Let E be a set, and let A and B be two subsets of E. Prove the following properties

$$1.\quad \varphi_A + \varphi_{C_E^A} = 1 \qquad 2.\quad \varphi_{A\cap B} = \varphi_A.\varphi_B \qquad 3.\ \varphi_{A\setminus B} = \varphi_A(1-\varphi_B).$$

where φ_A denotes the indicator mapping of the subset A, defined by

$$\begin{array}{rcll} \varphi_A : & E & \longrightarrow & \{0,1\} \\ & x & \longmapsto & \varphi_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases} \end{array}$$

Exercise n°4 :
1. Let $f : E \longrightarrow F$ be a mapping. Prove the following
(a) $\forall A,B \in \mathcal{P}(E) : A \subset B \Longrightarrow f(A) \subset f(B)$.
(b) $\forall A,B \in \mathcal{P}(E) : f(A\cap B) \subset f(A) \cap f(B)$.
(d) $\forall C,D \in \mathcal{P}(F) : C \subset D \Longrightarrow f^{-1}(C) \subset f^{-1}(D)$.
(e) $\forall C,D \in \mathcal{P}(F) : f^{-1}(C\cup D) = f^{-1}(C) \cup f^{-1}(D)$.
2. Let the mapping $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = x - y$, and let $A = \{0,1\}\times\{1,2\}$.
a. Find $f(A)$. Deduce that f is not injective.

Exercise n°5 : Let $E = [0,1]$ and $F = [0,2]$ be intervals in $\mathbb{R}$. Let f and g be two mappings defined by

$$\begin{array}{rcll} f : & E & \longrightarrow & F \\ & x & \longmapsto & f(x) = 2 - x \end{array}, \qquad \begin{array}{rcll} g : & F & \longrightarrow & E \\ & x & \longmapsto & g(x) = (x-1)^2 \end{array}$$

1. Determine the mappings $f\circ g$ and $g \circ f$.
2. Find $f^{-1}(\{0\})$ and deduce that f is not surjective.
3. Prove that $g\circ f$ is bijective, and find $(g\circ f)^{-1}$.

Exercise n°6 : Prove that the mapping

$$\begin{array}{rcll} f : & (\mathbb{N}^*, |) & \longrightarrow & (\mathbb{N}^*, |) \\ & x & \longmapsto & f(x) = x^2 \end{array}$$

is strictly increasing with respect to the divisibility relation.

Exercise n°7 : (Supplementary Exercise)
Let E, F, and G be nonempty sets, and let $f : E \longrightarrow F$, $g : F \longrightarrow G$ be two mappings. Prove the following properties :
1. If f and g are injective, then $g\circ f$ is injective.
2. If f and g are surjective, then $g \circ f$ is surjective.
3. If f and g are bijective, then $g\circ f$ is bijective.
4. If $g\circ f$ is injective, then f is injective.
5. If $g \circ f$ is surjective, then g is surjective.