الجمهورية الجزائرية الديمقراطية الشعبية

Democratic and Popular Republic of Algeria

وزارة التعليم العالي والبحث العلمي

Ministry of Higher Education and Scientific Research

Power Series

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Academic Year: 2025/2026

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Chapter 1

Power series

In this chapter, we will study a power series which are special forms of the series of functions of real or complex variables. For this, x denotes a real variable and z a complex variable.

1.1 Real (or complex) power series

Definition 1.1.1. *A real (resp. complex) power series is any series of functions whose general term:*

$$f_n(x) = a_n x^n, \tag{1.1}$$

where $a_0, a_1, ..., a_n, ...$ are real numbers and $x \in \mathbb{R}$ (resp.

$$f_n(x) = a_n z^n, \tag{1.2}$$

where $a_0, a_1, ..., a_n, ...$ are complex numbers and $z \in \mathbb{C}$.)

To unify the presentation of the following results, we consider the case where $x \in \mathbb{R}$.

Lemma 1.1.1. *(Abel's Lemma) If the power series $\sum a_n x^n$ converges at the point $x_0 \neq 0$, then it converges absolutely for all $x \in \mathbb{R}$, such that $|x| < x_0$.*

Proof. Since the power series $\sum a_n x_0^n$ converges, its general term is bounded, there then exists $M > 0$, such that:

$$\text{for all } n \in \mathbb{N},\ \left|a_n x_0^n\right| \leq M. \tag{1.3}$$

For all $x \in \mathbb{R}$, such that $|x| < x_0$, we thus have:

$$\begin{aligned} |a_n x^n| &= \left|a_n x_0^n\right| \times \left|\frac{x}{x_0}\right|^n \\ &\leq M\left|\frac{x}{x_0}\right|^n, \end{aligned} \tag{1.4}$$

and $\left|\frac{x}{x_0}\right|^n$ is the general term of a convergent geometric series ($\left|\frac{x}{x_0}\right| < 1$); we deduce that the series $\sum a_n x^n$ converges absolutely. □

1.1.1 Radius of convergence of a power series

Theorem 1.1.1. *(theorem and definition) If a power series $\sum a_n x^n$ converges to the point $x_0 \neq 0$, then there exists a unique element $R \in \mathbb{R}_+ \cup \{+\infty\}$ verifying the following two conditions:*

1. For all $x \in \mathbb{R}$, such that $|x| < R$, the power series $\sum a_n x^n$ absolutely converges .

2. For all $x \in \mathbb{R}$, such that $|x| > R$, the power series $\sum a_n x^n$ diverges.

The number R is called the radius of convergence of the series, and the set $]-R, R[$ is called the interval(or domain) of convergence.

Proof. Suppose that there exists at least one real $x_0 \neq 0$, such that the series $\sum a_n x_0^n$ converges and one real x_1 such that the series $\sum a_n x_1^n$ diverges.

Since absolute convergence on $[0, R[$ implies convergence on $]-R, 0]$, and divergence on $]R, +\infty[$ implies divergence on $]-\infty, -R[$, we will study the nature of $\sum a_n x^n$ on $\mathbb{R}_+$.

Let us then consider the set D of positive reals defined by:

$$D = \left\{x \in \mathbb{R}_+, \sum a_n x^n \text{ converge.}\right\} \tag{1.5}$$

Since the series $\sum a_n x_0^n$ converges, D is therefore non-empty.

According to the relation (1.5), the set D is majorized, it therefore admits a non-zero upper bound $R = \sup_{x \in \mathbb{R}_+} D$.

1. Prove that for all $x \in \mathbb{R}_+$, such that $x < R$, the power seires $\sum a_n x^n$ converges absolutely.
The second property of the upper bound states that there exists $r = x_0$ between x and R, such that $\sum a_n x^n$ converges at the point x_0, so according to Abel, it is absolutely convergent for all $x \in \mathbb{R}_+$, such that $x < x_0$.
2. Let us now show that for all $x \in \mathbb{R}_+$, such that $x > R$, the power series $\sum a_n x^n$ diverges.
Suppose by contradiction that $\sum a_n x^n$ converges, and consider $y = \dfrac{R+x}{2}$. Since $0 < y < x$, Abel's lemma states that the series $\sum a_n y^n$ converges, y is therefore a point of convergence, that is $y \in D$. Consequently $y \leq R$, and this is false, because by construction $y = \dfrac{R+x}{2} > R$, and the series $\sum a_n x^n$ diverges. □

1.1.2 Cauchy-Hadamard rule

Theorem 1.1.2. *The radius of convergence of a power series $\sum a_n x^n$ is given by:*

$$R = \lim_{n\to+\infty} \left(\sqrt{|a_n|}\right)^{-1} \text{ (when this limit exists).} \tag{1.6}$$

Proof. It suffices to apply the Cauchy criterion on the series of functions $\sum |a_n x^n|$ □

Example 1.1.1. *The power series $\sum_{n\geq 1} (\dfrac{n+1}{n})^{n^2} x^n$*

1.1.3 D'Alembert's rule

Theorem 1.1.3. *The radius of convergence of a power series $\sum a_n x^n$ is given by:*

$$R = \lim_{n\to+\infty} \left(\left|\frac{a_{n+1}}{a_n}\right|\right)^{-1} \text{ (when this limit exists).} \tag{1.7}$$

Proof. It suffices to apply D'Alembert's criterion on the series of functions $\sum |a_n x^n|$ □

Example 1.1.2. *The power series $\sum_{n\geq 1} \dfrac{n!}{n^n} x^n$ has for the radius of convergence $R = e$.*

1.1.4 Normal convergence (Weierstrass rule)

Theorem 1.1.4. *Any power series $\sum a_n x^n$ converges normally in any compact contained in the domain of convergence $]-R, R[$ $(R > 0)$.*

Proof. Let $[-\alpha, \alpha] \subset]-R, R[$ $(\alpha > 0)$. In the segment $[-\alpha, \alpha]$, the series $\sum a_n x^n$ is bounded above in absolute value by the positive series $\sum |a_n| \alpha^n$, which is convergent, the series $\sum a_n x^n$ is therefore normally convergent. □

1.2 Properties of power series

1.2.1 Continuity of the sum of a power series

Theorem 1.2.1. *Let $\sum a_n x^n$ be a power series with a non-zero radius of convergence R; then the sum S of the series $\sum a_n x^n$ is a continuous function on any compact set contained in the domain of convergence $]-R, R[$.*

Proof. For all $n \in \mathbb{N}$, each function $f_n(x) = a^n x^n$ is continuous on $[-\alpha, \alpha]$ of $]-R, R[$ and the series $\sum a_n x^n$ converges uniformly on $[-\alpha, \alpha]$. By the property of the continuity of series of functions, the sum of the power series $\sum a_n x^n$ is a continuous function. □

Theorem 1.2.2. *(Abel's theorem) Let $\sum a_n x^n$ be a power series with radius of convergence $R \neq 0$. If this series converges for $x = R$ (resp. for $x = -R$), then this series is uniformly convergent on $[0, R]$ (resp. on $[-R, 0]$) and the sum S of this series is continuous to the left of $x = R$ (resp. to the right of $x = -R$), that is:*

$$\lim_{x \to R^-} \sum a_n x^n = \sum a_n R^n = S(R), \tag{1.8}$$

(resp.

$$\lim_{x \to -R^+} \sum a_n x^n = \sum a_n (-1)^n R^n = S(-R). \tag{1.9}$$

Proof. We demonstrate this in the case where the series converges for $x = R$. Consider the new power series $\sum_{n \geq 0} a_n R^n y^n$ of the variable $y \in [0, 1]$.

For $y=1$, the series becomes $\sum_{n\geq 0} a_n R^n$, which is convergent, so it is uniformly convergent.
Let us now assume that $y \in [0,1[$. We use the Abel transformation, we can write:

$$\sum_{n\geq 0} a_n R^n y^n = \sum_{n\geq 0} (a_0 + a_1 R + \ldots + a_n R^n)\left(y^n - y^{n+1}\right) \tag{1.10}$$

Let

$$g_n(y) = (a_0 + a_1 R + \ldots + a_n R^n)\left(y^n - y^{n+1}\right).$$

We just need to show that this series is uniformly convergent.
Indeed:

$$\begin{aligned} \left|g_n(y)\right| &= \left|(a_0 + a_1 R + \ldots + a_n R^n)\left(y^n - y^{n+1}\right)\right| \\ &\leq M(y^n - y^{n+1}), \end{aligned} \tag{1.11}$$

because $y^n - y^{n+1} \geq 0$ ((y^n) is decreasing) and the sequence of general term $\sum_{k=0}^{n} a_k R^k$ is bounded.
For all $y \in [0,1[$, the sequence of general term y^n converges uniformly to 0, hence according to the telescopic property, the series $\sum\limits_{n\geq 1}^{+\infty} (y^n - y^{n+1})$ is uniformly convergent. The comparison theorem therefore asserts the uniform convergence of the series $\sum\limits_{n\geq 0}^{+\infty} g_n$. It follows that the initial series $\sum\limits_{n\geq 0}^{+\infty} a_n R^n y^n$ is uniformly convergent on $[0,1[$.
We then deduce the uniform convergence of $\sum\limits_{n\geq 0}^{+\infty} a_n R^n y^n$ on $[0,1]$.
Since each function $a_n R^n y^n$ is continuous on $[0,1]$, it results in the continuity of the sum of this series on $[0,1]$, and moreover:

$$\sum_{n\geq 0}^{+\infty} a_n R^n y^n = \begin{cases} S(yR), \text{ if } y \in [0,1[\\ \sum_{n\geq 0} a_n R^n, \text{ if } y = 1. \end{cases} \tag{1.12}$$

The continuity on the left at $y=1$ then gives us:

$$\lim_{y\to 1^-} S(yR) = S(R) = \sum_{n\geq 0} a_n R^n \tag{1.13}$$

□

1.2.2 Derivability of power series

Theorem 1.2.3. *Let $\sum\limits_{n\geq 0}^{+\infty} a_n x^n$ be a power series of non-zero radius of convergence R, then its sum S is a function derivable on any compact $[a,b]$ contained in the domain of convergence $]-R,R[$, and for any $x \in [a,b]$, we have:*

$$\dot{S}(x) = \frac{\partial}{\partial x}\left(\sum_{n\geq 0}^{+\infty} a_n x^n\right) = \sum_{n\geq 0}^{+\infty} \frac{\partial}{\partial x}(a_n x^n) = \sum_{n\geq 1}^{+\infty} n a_n x^{n-1}. \tag{1.14}$$

Proof. It suffices to show that the power series $\sum\limits_{n\geq 0}^{+\infty} a_n x^n$ and its derivative series $\sum\limits_{n\geq 1}^{+\infty} n a_n x^n$ have the same radius of convergence R; then the theorem of derivation of series of functions applies since a power series converges uniformly on any compact contained in the domain of convergence. Indeed, let R be the radius of convergence of the series $\sum(n+1)a_{n+1}x^n$.

1. If $|x| < R$, the series $\sum(n+1)a_{n+1}x^n$ is convergent.

Since:

$$\left|a_{n+1}x^{n+1}\right| \leq \left|(n+1)a_{n+}x^{n+1}\right| = (n+1)\left|a_{n+}x^n\right||x|, \tag{1.15}$$

the series $\sum\left|a_{n+}x^{n+1}\right|$ is convergent, the series $\sum a_{n+1}x^{n+1}$(or simply $\sum a_n x^n$) is therefore convergent.

2. If $|x| > R$, the series $\sum(n+1)a_{n+1}x^n$ is divergent. Let $y = \dfrac{R+|x|}{2} \in]R, |x|[$, the series $\sum(n+1)a_{n+1}y^n$ diverges and its general term is not bounded.

We can write:

$$\left|a_{n+1}x^{n+1}\right| = \left|(n+1)a_{n+1}y^n\right| \frac{1}{n+1}\left(\frac{|x|}{y}\right)^n. \tag{1.16}$$

Since $\dfrac{|x|}{y} > 1$, $\lim_{n\to+\infty} \frac{1}{n+1}\left(\dfrac{|x|}{y}\right)^n = +\infty$, so $\lim_{n\to+\infty}\left|a_{n+1}x^{n+1}\right| \neq 0$.

The series $\sum a_{n+1}x^{n+1}$(or simply $\sum a_n x^n$) is therefore divergent. □

Corollary 1.2.1. *Let $\sum\limits_{n\geq 0}^{+\infty} a_n x^n$ be a power series of non-zero radius of convergence R, then its sum S is an infinitely derivable function on any compact contained in the domain of convergence $]-R,R[$, and for any $x \in]-R,R[$ and $k \geq 1$ we have:*

$$S^{(k)}(x) = \sum_{n\geq 0}^{+\infty}(n+k)(n+k-1)...(n+1)a_{n+k}x^n. \tag{1.17}$$

Proof. It suffices to show by recurrence that the series $\sum\limits_{n\geq 0}^{+\infty}(n+k)(n+k-1)...(n+1)a_{n+k}x^n$, $k = 1, 2, ...$ have the same radius of convergence R. □

1.2.3 Integration of a power series

Theorem 1.2.4. *Any power series $\sum\limits_{n\geq 0}^{+\infty} a_n x^n$ is term by term integrable on any compact contained in the domain of convergence $]-R, R[$. In particular, its sum S verifies:*

$$\int_0^x S(t)dx = \sum_{n\geq 0}^{+\infty} a_n \frac{x^{n+1}}{n+1}, \textit{for all } x \in \,]-R, R[\,. \tag{1.18}$$

Proof. Let $x \in \,]-R, R[$. Since the power series $\sum\limits_{n\geq 0}^{+\infty} a_n x^n$ converges uniformly on $[0, x]$, the sum S is then a continuous function, and therefore integrable on $[0, x]$. The Equ. (1.18) is therefore well defined. Moreover, if we derivate the series (1.18), we find:

$$S(x) = \sum_{n\geq 0}^{+\infty} a_n x^n, \text{ for all } x \in \,]-R, R[\,. \tag{1.19}$$

The two power series $\sum\limits_{n\geq 0}^{+\infty} a_n x^n$ and $\sum\limits_{n\geq 1}^{+\infty} n a_n x^n$ then have the same convergence radius R . □

Example 1.2.1. *Consider the power series of general term:*

$$a_n x^n = \frac{x^n}{n},\ n \geq 1. \tag{1.20}$$

The d'Alembert criterion shows that this series is absolutely convergent on $]-1, 1[$ and has the sum S.
For all $x \in \,]-1, 1[$, the series $\sum\limits_{n\geq 1}^{+\infty} a_n x^n$ is derivable term by term. We then have:

$$\dot{S}(x) = \sum_{n\geq 0}^{+\infty} x^n = \frac{1}{1-x}, \textit{ for all } x \in \,]-1, 1[\,. \tag{1.21}$$

S is continuous on $[0, x]$, so it is integrable on this interval. We then have:

$$S(x) = -\ln(1-x). \tag{1.22}$$

On the other hand, if $x = -1$, the numerical series $\sum\limits_{n\geq 1}^{+\infty} \frac{(-1)^n}{n}$ converges, we can then apply Abel's theorem 1.2.2, we deduce that:

$$\sum_{n\geq 1}^{+\infty} \frac{(-1)^n}{n} = -\ln 2. \tag{1.23}$$

1.3 Sums and products of power series

1.3.1 Sum of two power series

Let $\sum a_n x^n$ and $\sum b_n x^n$ be two power series with radius of convergence R_a and R_b respectively, we then have:

Proposition 1.3.1. *The radius of convergence R of the power series $\sum(a_n + b_n)x^n$ verifies:*

$$\begin{aligned} R &\geq \inf(R_a, R_b), \text{ if } R_a = R_b. \qquad (1.24)\\ R &= \inf(R_a, R_b), \text{ if } R_a \neq R_b \end{aligned}$$

Moreover, for all $|x| < \inf(R_a, R_b)$, we have:

$$\sum(a_n + b_n)x^n = \sum a_n x^n + \sum b_n x^n. \tag{1.25}$$

Proof. 1. When $|x| < \inf(R_a, R_b)$, the two series $\sum a_n x^n$ and $\sum b_n x^n$ are convergent, the series $\sum(a_n + b_n)x^n$ is therefore convergent, we deduce that:

$$R \geq \inf(R_a, R_b). \tag{1.26}$$

Let $R_a \neq R_b$. Suppose for example that $R_a < R_b$, and let $x \in \mathbb{R}$, such that $R_a < |x| < R_b$, the series $\sum a_n x^n$ is therefore divergent while the series $\sum b_n x^n$ is convergent. The series $\sum(a_n + b_n)x^n$ is then divergent, and moreover:

$$R \leq R_a = \inf(R_a, R_b). \tag{1.27}$$

From (1.26) and (1.27), we deduce that $R = \inf(R_a, R_b)$.

2. If $R_a = R_b$, we cannot conclude anything about the radius of convergence

of the series $\sum(a_n + b_n)x^n$.
As an example, the two power series $\sum\left(\frac{n}{n+1}\right)^n x^n$ and $-\sum\left(\frac{n}{n+1}\right)^n x^n$ have the same radius of convergence $R_a = R_b = e$, while the sum series is the series with a zero general term, and therefore $R = +\infty$. □

1.3.2 Product of two power series

Proposition 1.3.2. *The radius of convergence R of the series with general term:*

$$c_n x^n = \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n \tag{1.28}$$

verifies $R \geq \inf(R_a, R_b)$. In addition, for all $|x| < \inf(R_a, R_b)$, we have:

$$\sum_{n\geq 0} c_n x^n = \left(\sum_{n\geq 0} a_n x^n\right) \times \left(\sum_{n\geq 0} b_n x^n\right). \tag{1.29}$$

Proof. let $|x| < \inf(R_a, R_b)$. Since the two series $(\sum_{n\geq 0} a_n x^n)$ and $(\sum_{n\geq 0} b_n x^n)$ are absolutely convergent, the Cauchy product series of general term:

$$\sum_{k=0}^{n} \left(a_k x^k\right)\left(b_{n-k} x^{n-k}\right) = \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n,$$

is also absolutely convergent, and for all $|x| < \inf(R_a, R_b)$, we have:

$$\sum_{n\geq 0} c_n x^n = \sum_{n\geq 0}\left[\sum_{k=0}^{n} \left(a_k x^k\right)\left(b_{n-k} x^{n-k}\right)\right] = \left(\sum_{n\geq 0} a_n x^n\right) \times \left(\sum_{n\geq 0} b_n x^n\right). \tag{1.30}$$

□

1.4 Functions developable in a power series (Taylor series)

In this section, we will study the problem in reverse.

1.4.1 Functions developable in a power series

Definition 1.4.1. *Let x_0 be a given real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined in the neighborhood of x_0. We say that f is developable in a power series at the point x_0, if there exists a power series $\sum_{n\geq 0} a_n x^n$ with radius of convergence $R > 0$, such that:*

$$\text{for all } x \in \mathbb{R}, |x - x_0| < R, \ f(x) = \sum_{n\geq 0} a_n (x - x_0)^n \tag{1.31}$$

By performing the change of variable $X = x - x_0$, we then speak of a function that is developable in a power series at the origin.

Definition 1.4.2. *A function f of a complex variable z is said to be developable in a power series at the point z_0, if there exists a power series $\sum_{n\geq 0} a_n (z - z_0)^n$, with radius of convergence $R > 0$, such that:*

$$\text{for all } z \in \mathbb{C}, |z - z_0| < R, \ f(z) = \sum_{n\geq 0} a_n (z - z_0)^n. \tag{1.32}$$

Definition 1.4.3. *If f is indefinitely differentiable, the power series with general term $\dfrac{f^{(k)}(0)}{k!} x^n$ is called Taylor series of f.*

1.4.2 Necessary condition for development in power series

Theorem 1.4.1. *When a function f is developable in power series, then f is of class $C^{+\infty}$ on any compact contained in the domain of convergence $]-R, R[$ and f coincides with its Taylor series. Moreover, if the power series development exists, it is unique.*

Proof. Suppose that f is developable in a power series at the origin, then there exists a power series $\sum_{n\geq 0} a_n x^n$ with a non-zero radius of convergence R, such that:

$$\text{for all } x \in \mathbb{R}, |x| < R, \text{ we have } f(x) = \sum_{n\geq 0} a_n x^n = S(x), \tag{1.33}$$

where S is the sum of this series.
According to the theorem of the derivation of power series, we deduce that f is of class $C^{+\infty}$ on $]-R, R[$ and moreover:

$$f^{(k)}(x) = \sum_{n\geq 0}^{+\infty}(n+k)(n+k-1)...(n+1)a_{n+k}x^n. \tag{1.34}$$

It follows that:

$$f^{(k)}(0) = a_k k!, \text{ i.e. } a_k = \frac{f^{(k)}(0)}{k!}, \tag{1.35}$$

which ensures the uniqueness of the development. □

Remak 1.1. *The converse of the previous theorem is false. Indeed, the condition that f is of class $C^{+\infty}$ on any compact contained in the domain of convergence $]-R, R[$, is not sufficient to ensure that this function is developable in a power series, even if its Taylor series converges. As an example, we consider the function f defined on $\mathbb{R}$ by:*

$$f(x) = \begin{cases} \exp(-\frac{1}{x^2}), \text{ if } x > 0, \\ 0, \text{ if } x \leq 0. \end{cases} \tag{1.36}$$

By recurrence, we can easily verify that this function is of class $C^{+\infty}$ on $\mathbb{R}$. Moreover for all $k \in \mathbb{N}$, the derivative of order k of f at point 0 is zero. So, if we assume that f is developable in a power series, its development is the zero series, which is impossible since $f(x) \neq 0$, for all $x \in]-R, R[$.

1.4.3 Sufficient condition for development in power series

Theorem 1.4.2. *Let f be an indefinitely derivable function on an interval $]-r, +r[$. A sufficient condition for f to be developable in a power series is the following:*

$$\exists M > 0, \ \forall n \in \mathbb{N}, \forall x \in]-r, +r[, \ \left|f^{(n)}(x)\right| \leq M. \tag{1.37}$$

In addition, for all $x \in]-r, +r[$, we have:

$$f(x) = \sum_{n\geq 0}^{+\infty} \frac{f^{(n)}(0)}{n!}x^n. \tag{1.38}$$

Proof. Since f is indefinitely derivable on $]-r,+r[$, the formula of Mac-Laurin gives:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{n!} x^k + \frac{x^{n+1}}{(+1)!} f^{(n+1)}(\theta x),\ \theta \in \,]0,1[\,. \tag{1.39}$$

It is enough to show that $\lim_{n\to+\infty} \frac{x^{n+1}}{(+1)!} f^{(n+1)}(\theta x) = 0$. Indeed, by hypothesis, we can write:

$$0 \leq \left| \frac{x^{n+1}}{(+1)!} f^{(n+1)}(\theta x) \right| \leq M \left| \frac{x^{n+1}}{(n+1)!} \right|. \tag{1.40}$$

Since $\frac{x^{n+1}}{(n+1)!}$ is the general term of a convergent series, we therefore have:

$$\lim_{n\to+\infty} \frac{x^{n+1}}{(n+1)!} = 0. \tag{1.41}$$

Consequently:

$$\lim_{n\to+\infty} \frac{x^{n+1}}{(+1)!} f^{(n+1)}(\theta x) = 0. \tag{1.42}$$

The function f is indeed the sum of its Taylor series on $]-r,+r[$. □

1.5 Development in power series of usual functions

1.5.1 The sine and cosine functions

These two functions are of class $C^{+\infty}$ on $\mathbb{R}$. By recurrence, we can easily verify that their n th derivatives are:

$$\sin^{(n)}(x) = \sin(x + n\frac{\pi}{2}) \tag{1.43}$$

$$\cos^{(n)}(x) = \cos(x + n\frac{\pi}{2}), \tag{1.44}$$

are indeed majored by $M = 1$, for all $x \in \mathbb{R}$. They are therefore developable in power series on $\mathbb{R}$, which means that $R = +\infty$. We therefore have:

$$\sin(x) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} x^{2n+1},\ \text{for all } x \in \mathbb{R}. \tag{1.45}$$

$$\cos(x) = \sum_{n\geq 0} \frac{(-1)^n}{(2n)!} x^{2n},\ \text{for all } x \in \mathbb{R}. \tag{1.46}$$

1.5.2 The exponential function $x \mapsto \exp(x)$

This function is of class $C^{+\infty}$ on any interval $]-r, r[$. By recurrence, we can easily verify that its n th derivative is also equal to $\exp(x)$, and is well bounded above $\exp(r)$. It is therefore developable in a power series on any interval $]-r, r[$. Since r is arbitrary, we deduce that $R = +\infty$. We therefore have:

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}, \text{ for all } x \in \mathbb{R}. \tag{1.47}$$

1.5.3 The logarithm function $x \mapsto \ln(1-x)$

The function $x \mapsto \dfrac{1}{1-x}$ is developable in a power series on $]-1, 1[$. Indeed, let $(x^n)_{n\in\mathbb{N}}$ be a geometric sequence, we can write:

$$\frac{1}{1-x} = \sum_{k=0}^{n} x^k + \frac{x^{n+1}}{1-x}, x \in \mathbb{R}/\{1\}. \tag{1.48}$$

We deduce that:

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n, x \in]-1, 1[. \tag{1.49}$$

Integrating term by term, we obtain:

$$\ln(1-x) = -\sum_{n\geq 0} \frac{x^{n+1}}{n+1}, x \in [-1, 1[\text{ (because } \sum_{n\geq 0} \frac{(-1)^{n+1}}{n+1} \text{ converges).} \tag{1.50}$$

Remak 1.2. *The techniques of the previous parts can be applied to obtain other developments from these cases. These techniques are adapted to the following functions:*

$$\cosh(x) = \frac{\exp(x) + \exp(-x)}{2} = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!}, \; x \in \mathbb{R}. \tag{1.51}$$

$$\sinh(x) = \frac{\exp(x) - \exp(-x)}{2} = \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!}, \; x \in \mathbb{R}. \tag{1.52}$$

$$\frac{1}{ax+b} = \frac{1}{b}\frac{1}{1-(-\frac{a}{b}x)} = \frac{1}{b}\sum_{n\geq 0}(-1)^n(\frac{a}{b})^n x^n, \; x \in \left]-\left|\frac{b}{a}\right|, \left|\frac{b}{a}\right|\right[\text{ and } a, b \neq 0. \tag{1.53}$$

$$\frac{1}{1+x^2} = \frac{1}{1+(-x^2)} = \sum_{n\geq 0}(-1)^n x^{2n}, \; x \in]-1, 1[. \tag{1.54}$$

$$\arctan(x) = \int_0^x \frac{dt}{1+t^2} = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{2n+1}, \, x \in [-1,1]. \tag{1.55}$$

$$\arg th(x) = \frac{1}{2}\ln(\frac{1+x}{1-x}) = \sum_{n\geq 0} \frac{x^{2n+1}}{2n+1}, \, x \in [-1,1[. \tag{1.56}$$

1.5.4 Rational functions

The decomposition of a rational function into simple elements and the use of the power series development of the function $x \mapsto f(x) = \frac{1}{1-x}$, allow us to develop a rational function in a power series.

Example 1.5.1. *We consider the rational function* $f(x) = \frac{1}{2-x}$.
We then have

$$f(x) = -\frac{1}{2} \times \frac{1}{1-\frac{x}{2}} = -\frac{1}{2}\sum_{n\geq 0}\left(\frac{x}{2}\right)^n, \, x \in]-2,2[. \tag{1.57}$$

1.6 Application to the resolution of certain differential equations

We will present here an example of a differential equation, a method that allows us to find a solution in the form of a function that can be developed in a power series over a certain interval $]-r,r[$.
Let us then consider the differential equation:

$$2x\dot{y} + y - \frac{1}{1-x} = 0. \tag{1.58}$$

Suppose there exists a power series $y(x) = \sum_{n\geq 0} a_n x^n$ with radius of convergence $r > 0$.
For all $x \in]-1,1[$,

$$\frac{1}{1-x} = 1 + \sum_{n\geq 1} x^n. \tag{1.59}$$

So, we have:

$$2x\dot{y} + y - \frac{1}{1-x} = a_0 + \sum_{n\geq 0}(2n+1)a_n x^n - \left(1 + \sum_{n\geq 1} x^n\right) = 0. \tag{1.60}$$

We deduce that:

$$a_0 = 1 \text{ and } a_n = \frac{1}{2n+1}. \tag{1.61}$$

Therefore:

$$y(x) = \sum_{n\geq 0} \frac{x^n}{2n+1}, \quad]-1,1[\tag{1.62}$$

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