Chapter 3: Damped Free Systems with One Degree of Freedom

Damped free vibrations are vibrations that occur after an initial disturbance, where the amplitude decreases over time until the motion eventually stops. The decrease in amplitude is due to the loss of energy caused by frictional forces. The friction is viscous and depends on the velocity.

$$f=-cv$$

where :

C : damping coefficient

V : velocity

Lagrange's Equation for a Damped System

In the case of a damped system, there exists a frictional force of the form

$$f=-c\dot{q}$$

and the energy loss is defined by the **dissipation function**:

$$D=\frac{1}{2}c\dot{q}^2$$

The equation of motion for a damped free system is therefore :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}+\frac{\partial D}{\partial \dot{q}}=0$$

The dissipation function is defined as:

$$D=\frac{1}{2}\alpha\dot{q}^2 \Longrightarrow \frac{\partial D}{\partial \dot{q}}=\alpha\dot{q}$$

The differential equation of motion takes the form:

$$\ddot{q}+2\delta\dot{q}+\omega_0 2q=0$$

where :

δ is the damping coefficient.

ω_0 is the undamped natural angular frequency..

Example: In the case of a **mass–spring system**, we have:

The **kinetic energy** of the mass: $T = \frac{1}{2} m\dot{x}^2$

The **potential energy** of the spring: $U = \frac{1}{2} kx^2$

The dissipation fonction $D = \frac{1}{2} \alpha \dot{q}^2$

Therfore :

$d/dt\ (\partial L/\partial \dot{x}) = m\ddot{x}$ and $\partial L/\partial x = kx$ and $\partial D/\partial \dot{x} = \alpha\dot{x}$

The equation of motion is then : $m\ddot{x} + kx + \alpha\dot{x} = 0 \Rightarrow$

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$

This is a **second-order linear differential equation**.

More generally, for a **generalized coordinate** q, it can be written as:

$$\mathbf{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0}$$

Solution of the Differential Equation

The second-order linear differential equation:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0$$

has the following **characteristic equation**:

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0$$

Depending on the nature of the roots of this characteristic equation, there are **three types of damping**:

$\acute{\Delta} < 0 \Rightarrow \delta^2 - \omega_0^2 < 0$ Weakly damped regime.

$\acute{\Delta} = 0 \Rightarrow \delta^2 - \omega_0^2 = 0$ Critical damping regime.

$\grave{\Delta}>0 \Rightarrow \delta^2 - \omega_0{}^2 > 0$ Strongly damped or aperiodic regime.

The motion is **aperiodic (overdamped)** when $\delta>\omega_0$, and the **solution** is of the form:

$$q(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

A1 and A2 are **integration constants** determined by the **initial conditions**.

Figure II-2 shows the solution q(t) as a function of time in the particular case where:

$q(0)=q_0$ and $\dot{q}(0)=0$. In this case, the solution q(t) **decreases exponentially toward zero**,

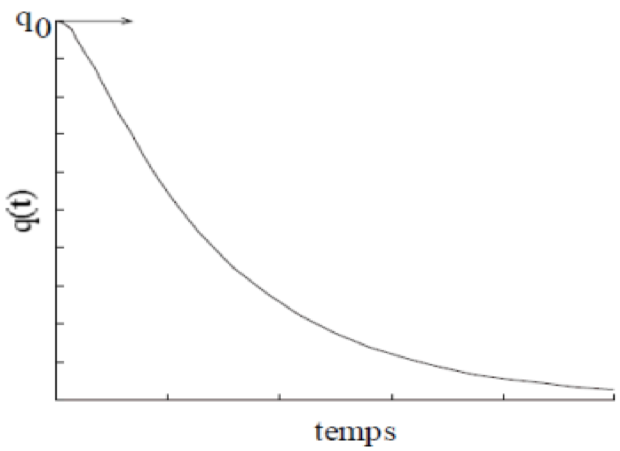


Fig. III-1. Variation of q(t) as a function of time for the **overdamped regime**.

When $\delta=\omega_0$, the system is in the **critically damped regime** (see **Figure II-3**), and the solution takes the form:

$$q(t) = (A1 + A2t)^{-\delta t}$$

In this case, the motion returns to equilibrium **as quickly as possible without oscillating**. The displacement q(t) decreases **monotonically** toward zero, and the system does not overshoot its equilibrium position.

The constants A1 and A2 are determined by the initial conditions q(0) and $\dot{q}(0)$.

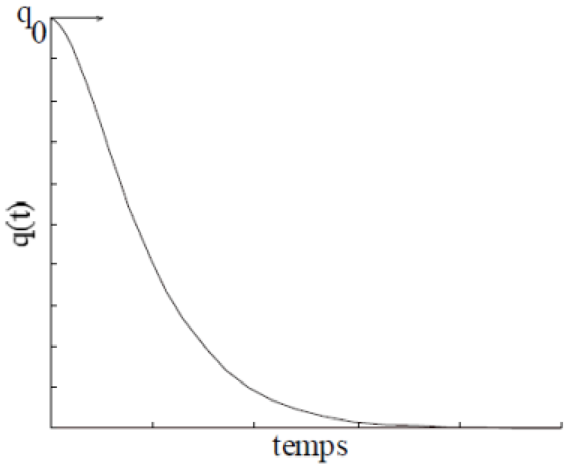


Fig. III-2. Variation of q(t) as a function of time for the **critically damped regime**.

When $\delta<\omega0$, the system is in the **underdamped (pseudoperiodic) regime** (see **Figure III-3**), and the solution takes the form:

$$q(t) = Ae^{-\delta t}co(\omega_a t + \varphi)$$

A and φ are **integration constants** determined from the **initial conditions**.

ωa\omega_aωa is the **damped (pseudo) angular frequency**, defined by:

$$\omega a = \sqrt{\omega_0^2 - \delta^2}.$$

Figure III-3 illustrates the variation of q(t) as a function of time. It can be observed that q(t) is enveloped by two exponential functions. The locations of the maxima are obtained by solving:

$\dot{q}(t)$ =0. The maxima of q(t) are separated by regular intervals equal to Ta.

Ta is called the pseudo-period. It can be noted that the amplitude of the oscillations decreases over time, and one of the effects of damping is an increase in the oscillation period**.**

For lightly damped systems ($\delta << \omega0$), we can approximate:

$$\omega a \approx \omega_0$$

and the pseudo-period is almost equal to the natural period:

$$Ta \approx T_0 = 2\pi/\omega_0.$$

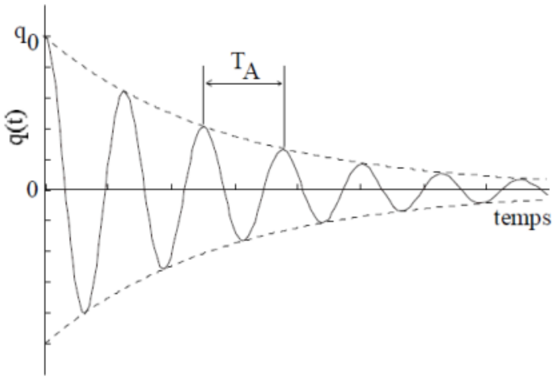


Fig. III-3. Variation of q(t)q(t)q(t) as a function of time for the lightly damped (underdamped) regime**.**

- **Critical damping coefficient**

Cc is the value of the damping coefficient C corresponding to $\Delta=0$, that is:

$$\left(\frac{Cc}{m}\right)^2 = 4\,k\,m \Rightarrow$$

$$\mathbf{Cc = 2m\sqrt{\frac{k}{m}} = \ 2m\omega_0}$$

This is the **minimum damping** required for the system to return to equilibrium without oscillating**.**

- **Damping ratio**

The **damping ratio** is defined as:

$$\xi = \frac{c}{Cc} \Rightarrow \frac{C}{2m} \ = \varepsilon\omega_0$$

$$\Rightarrow \xi = \frac{C}{2m\omega 0}$$

The damping ratio ζ characterizes the type of damping in the system:

- $\zeta=0$: undamped
- $0<\zeta<1$: underdamped (pseudoperiodic)
- $\zeta=1$: critically damped
- $\zeta>1$: overdamped (aperiodic)

- **Quality factor**

The **quality factor** is defined as:

$$Q = 2\pi\frac{E}{\Delta E} = \omega_0\, 2\delta$$

where:

- E is the energy of the harmonic oscillator,
- ΔE is the energy dissipated during one cycle,
- ω_0 is the undamped natural angular frequency,
- δ is the damping coefficient.

The smaller the damping, the higher the quality of the system. A high Q indicates that the system oscillates for a long time before its energy is significantly dissipated.

- **Logarithmic decrement (D)**

Figure II-4 illustrates the definition of the logarithmic decrement. It is defined as the natural logarithm of the ratio of two successive amplitudes of the damped oscillations:

$$D = ln\frac{A(t1)}{A(t2)} = ln\frac{A(t1)}{A(t1+Ta)} = -ln\frac{A(t1+Ta)}{A(t1)}$$

By substituting the expressions for the amplitudes, we obtain:

$$D = \delta Ta$$

where:

- δ is the damping coefficient,
- Ta is the pseudo-period, given by: $Ta = 2\pi/\omega a$

and ωa is the damped (pseudo) angular frequency, defined as:

$$\omega a = \sqrt{\omega 02 - \delta 2}$$

This shows that the logarithmic decrement is directly proportional to the damping coefficient and the pseudo-period of the system.

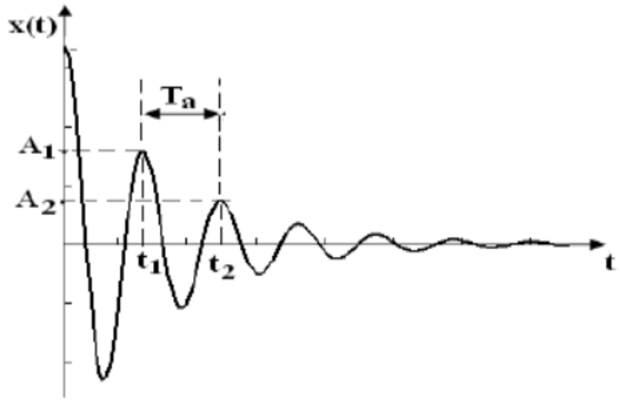
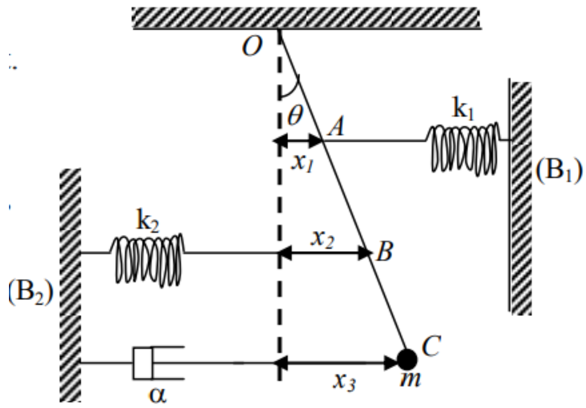


Fig. III-4. Definition of the logarithmic decrement

Exercise



A mass m is welded to the end of a rod of length l and negligible mass (Figure III-5). The other end of the rod is hinged at point O. The rod is connected at point A to a frame (B1) by a spring of stiffness k1. At point B, the rod is connected to another frame (B2) by a spring of stiffness k2. The mass m is connected to frame B2 by a **damper** with damping coefficient α. The distances are OA=l/3 and OB=2l/3.

Tasks:

1. Find the **differential equation of motion**.
2. Determine the **solution of the differential equation** in the case of **light damping**, including:
   - the damping coefficient,
   - the natural frequency,
   - and the damped (pseudo) frequency.