

ABDELHAFID BOUSSOUF UNIVERSITY CENTER, MILA
INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE
DEPARTMENT OF COMPUTER SCIENCE

Algebra I: Course Notes.

Chapter 3. Functions and Mappings

Author
Dr. Kecies Mohamed

Life is good only for two things, discovering mathematics and teaching mathematics.

SIMÉON DENIS POISSON

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Chapter 1

Notions of Logic

Chapter 2

Sets and Binary Relations

Chapter 3

Functions and Mappings

This chapter introduces the fundamental concepts of functions and mappings in mathematics. We begin with functions as relations between sets where each input is associated with at most one output. When this association is defined for every element in the starting set, we call it a mapping.

The chapter explores essential concepts including domain, image, and function notation, then moves to composition of mappings and the study of direct and inverse images of sets. We classify mappings into injective, surjective, and bijective types, examine inverse mappings, and conclude with the relationship between mappings and order relations.

Part I

Functions

3.1 Definitions and notations

Definition 1 Let E and F be two non-empty sets.

1. A function from the set E to the set F is a correspondence (or relation), denoted f , that associates to each element x in E **at most one element** y in F .
2. If an element $x \in E$ is associated with an element $y \in F$ by the function f , we write $y = f(x)$. In this case, y is called the *image* (or the *value*) of x under f , and x is called a *preimage* (or *antecedent*) of y under f .

Notation 2

- A function is written as

$$f : E \longrightarrow F$$

$$x \longmapsto y = f(x),$$

or, $f : x \longmapsto f(x)$.

- The notation $f : E \longrightarrow F$ is read as " f is a function from E to F " or " f is a function defined on E with values in F and $x \longmapsto f(x)$ is read as " x is associated with its image $f(x)$ ".

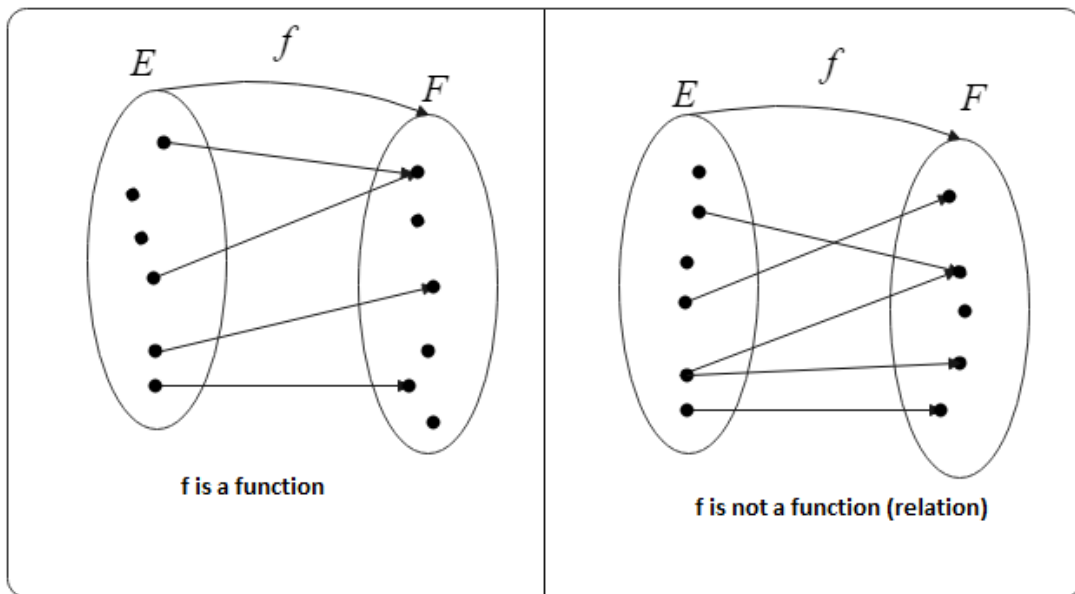


Figure 3.1: Function, not Function

Definition 3 Let E and F be two non-empty sets, and let $f : E \longrightarrow F$ be a function.

1. The domain (or the set of definition) of the function f , denoted $\text{Dom}(f)$, is the set

$$\text{Dom}(f) = \{x \in E, \exists y \in F : f(x) = y\} \subset E.$$

In other words, $\text{Dom}(f)$ is the set of elements in E that have an image under f , or equivalently, the set of all x for which $f(x)$ exists and has a value.

2. The image set of f , denoted $\text{Im}(f)$ or $f(E)$, is the set

$$\text{Im}(f) = \{y \in F, \exists x \in E : f(x) = y\} = \{f(x) \in F : x \in E\} \subset F.$$

In other words, $\text{Im}(f)$ is the set of all values taken by the function f .

Example 4

1. Consider the function (Absolute Value Function)

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = |x|, \end{aligned}$$

then:

- Domain of f : The absolute value function is defined for all real numbers, so $\text{Dom}(f) = \mathbb{R}$.
- Image set of f : For every real number x , $|x|$ is always non-negative, so $\text{Im}(f) = \mathbb{R}^+$.

2. Consider the function (Square Root Function)

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \sqrt{x}, \end{aligned}$$

- Domain of f : $f(x)$ is only exists for $x \geq 0$, so $\text{Dom}(f) = \mathbb{R}^+$.
- Image of f : Since $\sqrt{x} \geq 0$, the image is $\text{Im}(f) = \mathbb{R}^+$.

Part II

Mappings

3.2 Definitions and notations

Definition 5 Let E and F be two non-empty sets.

1. A mapping from a set E to a set F is a relation that associates each element $x \in E$ with a unique element, denoted by $f(x)$ in the set F .

• In other words: For every $x \in E$, there exists exactly one $y \in F$ such that (x, y) is in the relation.

2. Using quantifiers, this can be expressed as

$$f \text{ is a mapping} \iff \forall x \in E, \exists! y \in F : f(x) = y.$$

Remark 6 A function $f : E \longrightarrow F$ is a mapping if and only if $\text{Dom}(f) = E$, where $\text{Dom}(f)$ denotes the domain of f .

Notation 7 The set of all mappings $f : E \longrightarrow F$ is denoted by $\mathcal{F}(E, F)$ or F^E , and if $E = F$ it is denoted by $\mathcal{F}(E)$ or E^E .

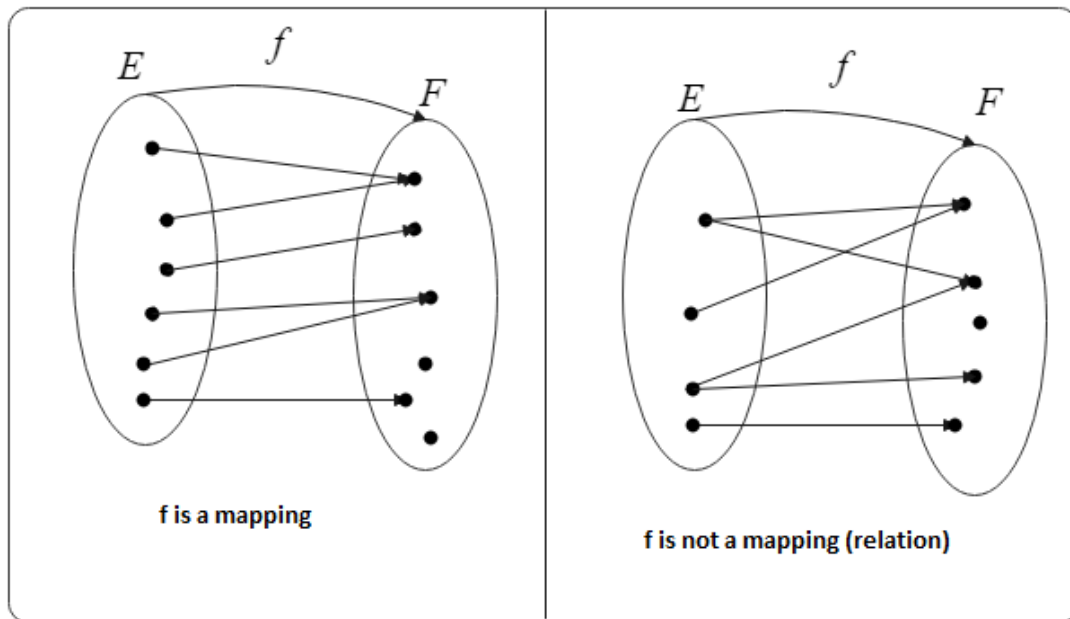


Figure 3.2: Mapping, not Mapping

Example 8 $f : \{1, 2, 3, 4\} \longrightarrow \{0, 1\}$ defined by

$$n \mapsto f(n) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

is a mapping because each element of $\{1, 2, 3, 4\}$ has exactly one image in $\{0, 1\}$.

$$f(1) = 1, \quad f(2) = 0, \quad f(3) = 1, \quad f(4) = 0.$$

Definition 9 (Equality of mappings)

Two mappings $f : E_1 \longrightarrow F_1$ and $g : E_2 \longrightarrow F_2$ are said to be equal if the following three properties hold:

- (i) $E_1 = E_2$ (same domain),
- (ii) $F_1 = F_2$ (same codomain),
- (iii) For all x , $f(x) = g(x)$.

In other words, two mappings are equal if they have the same domain and codomain and their values are the same for all elements of the domain.

Example 10 Consider the mappings

$$\begin{array}{ll} f_1 : \mathbb{R} \longrightarrow \mathbb{R} & \text{and} \quad f_2 : \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto f_1(x) = \cos x & x \longmapsto f_2(x) = 2 \cos^2\left(\frac{x}{2}\right) - 1. \end{array}$$

We have $f_1 = f_2$. Indeed, using the trigonometric identity

$$\cos 2\alpha = 2 \cos^2 \alpha - 1,$$

and setting $\alpha = \frac{x}{2}$, it follows that

$$f_1(x) = \cos x = 2 \cos^2\left(\frac{x}{2}\right) - 1 = f_2(x).$$

Then $f_1 = f_2$.

Definition 11 (Special mappings)

1. **Identity mapping:** The identity mapping of a set E is the mapping from E to E , denoted by Id_E (or sometimes just Id) defined as follows:

$$\forall x \in E : Id_E(x) = x.$$

2. **Constant mapping:** A mapping $f : E \longrightarrow F$ is called a constant mapping if there exists an element $c \in F$ such that

$$\forall x \in E : f(x) = c.$$

A constant mapping is often denoted simply by the constant value c (i.e, $f = c$).

3. Zero Mapping: Let E be a set, and let $f : E \longrightarrow \mathbb{R}$ be a mapping. The zero mapping, denoted by 0 , is defined by

$$\forall x \in E : f(x) = 0.$$

4. Indicator Mapping: Let A be a subset of a set E . The indicator mapping of A in E (sometimes referred to as the characteristic mapping of A in E), denoted by φ_A is the mapping defined as follows

$$\begin{array}{ccc} \varphi_A : E \longrightarrow \{0, 1\} & & \varphi_A : E \longrightarrow \{0, 1\} \\ x \longmapsto \varphi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} & \iff & x \longmapsto \varphi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in \overline{A}. \end{cases} \end{array}$$

Example 12

1. It is clear that

$$\varphi_{\mathbb{R}^+}(2) = 1, \varphi_{\mathbb{R}^-}(2) = 0, \varphi_{\mathbb{Q}}(\pi) = 0.$$

2. Consider the mapping

$$f(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

Using an indicator mapping, f can be written in a single line as

$$f(x) = e^{-x} \varphi_{[0, +\infty[}(x),$$

for all values of x .

Proposition 13 Let E be a set, and A and B be two subsets of E . Then the indicator mapping satisfies the following properties

- | | |
|---|---|
| 1. $\varphi_A^2 = \varphi_A$ | 2. $A \subset B \iff \varphi_A \leq \varphi_B$ |
| 3. $A = B \iff \varphi_A = \varphi_B$ | 4. $\varphi_{C_E^A} = 1 - \varphi_A$ |
| 5. $\varphi_{A \cap B} = \varphi_A \cdot \varphi_B$ | 6. $\varphi_{A B} = \varphi_A(1 - \varphi_B)$ |
| 7. $\varphi_{A \cup B} = \varphi_A + \varphi_B - \varphi_A \cdot \varphi_B$ | 8. $\varphi_{A \Delta B} = \varphi_A + \varphi_B - 2 \cdot \varphi_A \varphi_B = (\varphi_A - \varphi_B)^2 = \varphi_A - \varphi_B $. |

3.3 Extension and restriction of mapping

Definition 14 Let $f : E \longrightarrow F$ be a mapping, with $E_1 \subset E$ and $E \subset E_2$.

1. The restriction of f to E_1 , denoted $g = f|_{E_1}$, is the mapping from E_1 to F defined by

$$\forall x \in E_1 : g(x) = f|_{E_1}(x) = f(x).$$

In other words, g coincides with f on E_1 .

2. The extension of f to E_2 is any function $\tilde{f}$ from E_2 to F whose restriction to E is equal to f , i.e.,

$$\forall x \in E : \tilde{f}(x) = f(x).$$

Example 15 The mapping

$$\begin{aligned} \tilde{f} : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \tilde{f}(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0, \end{cases} \end{aligned}$$

is an extension of the mapping f to $\mathbb{R}$ such that

$$\begin{aligned} f : \mathbb{R}^* &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \frac{\sin x}{x}. \end{aligned}$$

and f is a restriction of $\tilde{f}$ to $\mathbb{R}^*$.

3.4 Composition of mappings

The composition of a mappings is an operation where two mappings say f and g generate a new function say h in such a way that $h(x) = g(f(x))$. It means here mapping g is applied to the mapping of x . So, basically, a mapping is applied to the result of another mapping.

Definition 16 Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be two mappings. The composition of g and f , denoted $g \circ f$ (read as "g round f" or "g composed with f"), is the mapping from E to G defined by

$$\forall x \in E, g \circ f(x) = g(f(x)).$$

Remark 17

1. The composition of two mappings can be represented schematically as follows

$$\begin{array}{ccccc} E & \xrightarrow{f} & F & \xrightarrow{g} & G \\ x & \mapsto & f(x) = y & \mapsto & g(y) \end{array}$$

$$\begin{array}{ccc} E & \xrightarrow{g \circ f} & G \\ x & \mapsto & g(f(x)) \end{array}$$

2. In general, $g \circ f \neq f \circ g$ where f and g are two mappings from $\mathcal{F}(E)$.

Example 18 Consider the two mappings

$$\begin{array}{ccc} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & f(x) = x + 1, \end{array}$$

and

$$\begin{array}{ccc} g : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & g(x) = x^2. \end{array}$$

For every real number x , we have

$$(g \circ f)(x) = g(f(x)) = (f(x))^2 = (x + 1)^2,$$

while

$$(f \circ g)(x) = f(g(x)) = g(x) + 1 = x^2 + 1.$$

In particular, $(g \circ f)(-1) = 0$ and $(f \circ g)(-1) = 2$. Therefore, here $g \circ f \neq f \circ g$.

Remark 19 Let $f : E \longrightarrow E$ be a mapping. We define

$$f^0 = Id_E.$$

For every $n \in \mathbb{N}^*$, the composition of f with itself n times is denoted by

$$f^n = \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}} = f \circ f^{n-1} = f^{n-1} \circ f.$$

Proposition 20 Let $f : E \longrightarrow F, g : F \longrightarrow G, h : G \longrightarrow H$ be mappings. Then we have

- The composition operation is associative

$$(h \circ g) \circ f = h \circ (g \circ f).$$

- The identity property

$$f \circ Id_E = Id_F \circ f = f.$$

3.5 Direct Image, Inverse Image of Sets

Definition 21 Let $f : E \longrightarrow F$ be a mapping. Let $A \subset E$ and $B \subset F$. Then

1. (a) The direct image of the subset A under f , denoted $f(A)$, is the set of all images of the elements of A under f . Formally

$$f(A) = \{y \in F, \exists x \in A : f(x) = y\} = \{f(x) : x \in A\} \subset F.$$

In terms of quantifiers, this can be written as

$$\forall y \in F : (y \in f(A) \iff \exists x \in A : f(x) = y).$$

- (b) The image $f(E)$ of the domain E , is called the image of f (or the range of f), denoted $Im(f)$,

$$Im(f) = f(E) = \{f(x) : x \in E\} \subset F.$$

2. The inverse image (or Preimage) of the subset B under f , denoted $f^{-1}(B)$, is the set of all preimages of the elements of B under f . Formally

$$f^{-1}(B) = \{x \in E, \exists y \in B : f(x) = y\} = \{x \in E : f(x) \in B\} \subset E.$$

Using quantifiers, this can be expressed as

$$\forall x \in E : (x \in f^{-1}(B) \iff f(x) \in B).$$

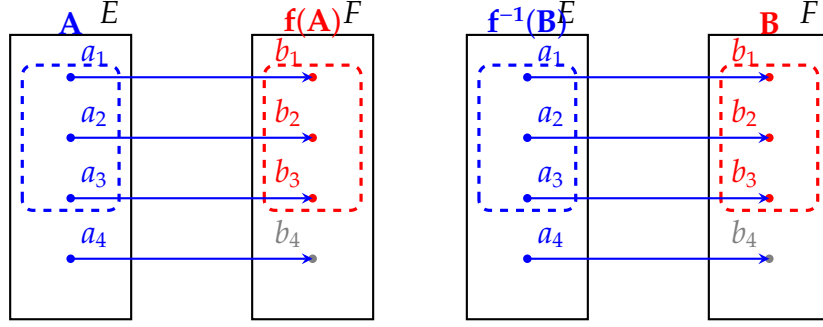


Figure 3.3: (left) Direct image $f(A)$, (right) Inverse image $f^{-1}(B)$

Example 22 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $x \longmapsto f(x) = x^2$, and let $A = [-1, 1]$. Then,

$$f(A) = f([-1, 1]) = \{x^2 : x \in [-1, 1]\} = [0, 1],$$

and

$$\text{Im}(f) = f(\mathbb{R}) = \mathbb{R}^+.$$

For $B = \{4\}$, we find that

$$f^{-1}(B) = \{-2, 2\}.$$

Indeed,

$$x \in f^{-1}(B) \iff f(x) \in B = \{4\} \iff f(x) = 4$$

Therefore, the preimage is $x \in \{-2, 2\}$.

Proposition 23 (Properties of Direct Image and Inverse Image)

Let $f : E \longrightarrow F$ be a mapping. Then, the following properties hold:

1. Properties of Direct Image:

(a). Monotonicity:

$$\forall A, B \in \mathcal{P}(E) : A \subset B \implies f(A) \subset f(B).$$

That is, the image of a subset of E is contained within the image of a larger subset.

(b). Image of Union

$$\forall A, B \in \mathcal{P}(E) : f(A \cup B) = f(A) \cup f(B).$$

The image of the union of two sets is the union of the images of those sets.

(c). Image of Intersection:

$$\forall A, B \in \mathcal{P}(E) : f(A \cap B) \subset f(A) \cap f(B).$$

The image of the intersection of two sets is contained within the intersection of their images.

(d) *Image of Complement*

$$\forall A \in \mathcal{P}(E) : f(C_E^A) \subset C_F^{f(A)}.$$

The image of the complement of a subset is a subset of the complement of the image.

(e). *Image of the entire Domain and the empty set:*

$$f(\phi) = \phi, f(E) \subset F.$$

2. Properties of Inverse Image:

(a). *Monotonicity:*

$$\forall C, D \in \mathcal{P}(F) : C \subset D \implies f^{-1}(C) \subset f^{-1}(D).$$

The preimage of a smaller subset is contained within the preimage of a larger subset.

(b). *Preimage of Union:*

$$\forall C, D \in \mathcal{P}(F) : f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D).$$

The preimage of the union of two sets is the union of their preimages.

(c). *Preimage of Intersection:*

$$\forall C, D \in \mathcal{P}(F) : f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D).$$

The preimage of the intersection of two sets is the intersection of their preimages.

(d) *Preimage of Complement:*

$$\forall D \in \mathcal{P}(F) : f^{-1}(C_F^D) = C_E^{f^{-1}(D)}.$$

The preimage of the complement of a set is the complement of the preimage.

(e) *Preimage of the empty set and the entire codomain:*

$$f^{-1}(\phi) = \phi, f^{-1}(F) = E.$$

3.6 Classification of Mappings

The mappings can be classified into different types based on their properties, particularly focusing on how the elements from the domain are mapped to the codomain. The three most common

classifications are injective, surjective, and bijective mappings.

3.6.1 Injective, Surjective, Bijective mappings

Definition 24 Let $f : E \longrightarrow F$ be a mapping. Then

1. *Injective mappings (One-to-One mappings):* We say that f is injective, or an injection from E to F , if and only if

$$\forall x_1, x_2 \in E : x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

Equivalently, f is injective if and only if

$$\forall x_1, x_2 \in E : f(x_1) = f(x_2) \implies x_1 = x_2.$$

2. *Surjective mappings (Onto mappings):* We say that f is surjective or a surjection from E to F if and only if

$$\forall y \in F, \exists x \in E : f(x) = y.$$

Equivalently, f is surjective if and only if

$$Im(f) = f(E) = F.$$

3. *Bijective mappings (One-to-One Correspondences):* We say that f is bijective (or a bijection from E to F) if and only if f is both injective and surjective. Equivalently, f is bijective if and only if

$$\forall y \in F, \exists! x \in E : f(x) = y.$$

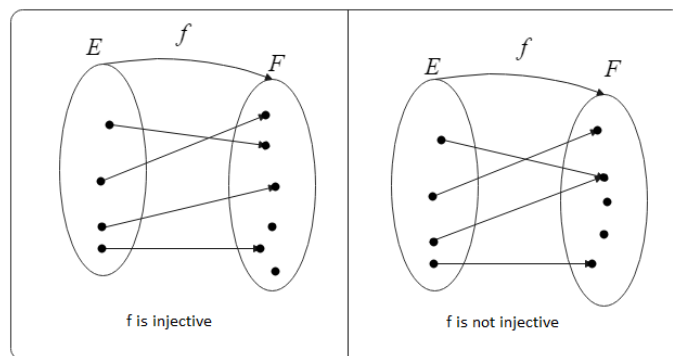


Figure 3.4: Injective mapping, not injective mapping

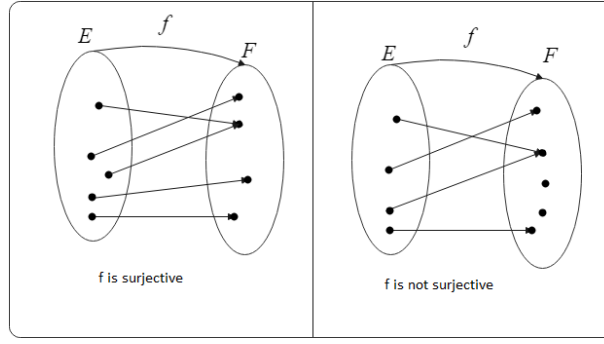


Figure 3.5: Surjective mapping, not surjective mapping

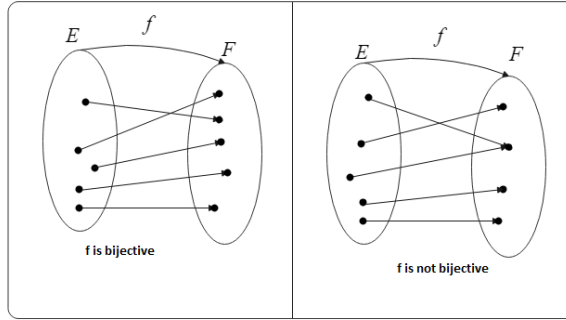


Figure 3.6: Bijective mapping, not bijective mapping

Remark 25 Let $f : E \longrightarrow F$ be a mapping. Then

1. f is injective if and only if every element $y \in F$ has at most one preimage $x \in E$ under f .
2. f is surjective if and only if every element $y \in F$ has at least one preimage $x \in E$ under f .
3. f is bijective if and only if every element $y \in F$ has exactly one preimage $x \in E$ under f .

Example 26

1. For any set E , the identity mapping Id_E is injective. Indeed, let $x_1, x_2 \in E$ such that $\text{Id}_E(x_1) = \text{Id}_E(x_2)$. By the definition of the Id_E , we have $x_1 = x_2$. Therefore, Id_E is injective.
2. Let $f : \mathbb{Z} \longrightarrow \mathbb{N}$ be defined by $f(x) = x^2$. Then, f is not surjective. Indeed, there exist elements $y \in \mathbb{N}$ that have no preimages, such as $y = 2$. Suppose 2 has a preimage x under f . Then, we would have

$$\begin{aligned} f(x) = 2 &\implies x^2 = 2 \\ &\implies x = \sqrt{2} \text{ or } x = -\sqrt{2}. \end{aligned}$$

However, neither $\sqrt{2}$ nor $-\sqrt{2}$ are integers in $\mathbb{Z}$. Therefore, $y = 2$ has no preimage, and f is not surjective.

3. The exponential mapping $f : x \longmapsto \exp(x)$ is a bijective mapping from $\mathbb{R}$ to $\mathbb{R}_+^*$. Specifically:

- *Injectivity:* If $\exp(x_1) = \exp(x_2)$, then $x_1 = x_2$, confirming that f is injective.
- *Surjectivity:* For every $y > 0$, there exists $x \in \mathbb{R}$ such that $y = \exp(x)$, with $x = \ln(y)$, confirming that f is surjective.

4. The natural logarithm mapping defined by

$$\begin{aligned} f : \mathbb{R}_+^* &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \ln x, \end{aligned}$$

is bijective. This is because it is both injective and surjective.

Proposition 27 (Composition of Injective, Surjective, and Bijective mappings)

Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be two mappings. The following properties hold:

1. If f and g are injective, then $g \circ f$ is injective.
2. If f and g are surjective, then $g \circ f$ is surjective.
3. If f and g are bijective, then $g \circ f$ is bijective.
4. If $g \circ f$ is injective, then f is injective.
5. If $g \circ f$ is surjective, then g is surjective.

There is a very special case where to prove the bijectivity of $f : E \longrightarrow F$, it is sufficient to show either its surjectivity or injectivity. This occurs when E and F are finite sets with the same cardinality.

Theorem 28 Let E and F be finite sets with the same cardinality ($\text{Card}(E) = \text{Card}(F)$). If $f : E \longrightarrow F$ is a mapping, then

$$f \text{ is bijective} \iff f \text{ is injective} \iff f \text{ is surjective}.$$

3.7 Inverse mappings

If f is a bijective mapping from a set E to a set F , then there is a mapping from F to E that “undoes” the action of F ; that is, it sends each element of F back to the element of E that it came from. This mapping is called the inverse mapping for f and is usually denoted by f^{-1} .

Definition 29 Let $f : E \longrightarrow F$ be a bijective mapping.

1. The mapping from F to E that associates each element $y \in F$ in the codomain of f with its **unique**

preimage $x \in E$, such that $y = f(x)$, is called the inverse mapping of f , denoted f^{-1} .

2. The inverse mapping f^{-1} is characterized by the following property

$$\forall (x, y) \in E \times F : y = f(x) \iff x = f^{-1}(y).$$

Example 30

1. The mapping

$$\begin{aligned} f : \mathbb{R} &\longrightarrow]0, +\infty[\\ x &\longmapsto f(x) = e^x \end{aligned}$$

is bijective. Its inverse is the mapping

$$\begin{aligned} f^{-1} :]0, +\infty[&\longrightarrow \mathbb{R} \\ x &\longmapsto f^{-1}(x) = \ln(x). \end{aligned}$$

because

$$\forall (x, y) \in \mathbb{R} \times]0, +\infty[: y = e^x \iff x = \ln(y).$$

2. Let E be a set. Then the mapping

$$\begin{aligned} f : \mathcal{P}(E) &\longrightarrow \mathcal{P}(E) \\ X &\longmapsto f(X) = C_E^X \end{aligned}$$

is bijective, and $f^{-1} = f$.

Proposition 31 (Characterization of a Bijection and Its Inverse)

Let $f : E \longrightarrow F$ be a mapping.

1. f is bijective if and only if there exists a mapping $g : F \longrightarrow E$ such that

$$g \circ f = \text{Id}_E \text{ and } f \circ g = \text{Id}_F.$$

where Id_E and Id_F denote the identity mappings on E and F , respectively.

2. If f is bijective, then the mapping g is unique and also bijective. Moreover,

$$g^{-1} = (f^{-1})^{-1} = f.$$

In other words,

$$g^{-1} = (f^{-1})^{-1} = f.$$

Corollary 32

1. Let $f : E \longrightarrow F$ be a bijective mapping. Then the inverse mapping f^{-1} of f satisfies

$$f^{-1} \circ f = Id_E \text{ and } f \circ f^{-1} = Id_F.$$

2. Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be two bijective mappings. Then the inverse of the composition $g \circ f$ is given by the following formula

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

3.8 Order and mappings

Definition 33 Let $(E, \leq)$ and $(F, \leq)$ be two ordered sets, and let $f : E \longrightarrow F$ be a mapping. Then we say that:

1. f is increasing if and only if

$$\forall x, y \in E : x \leq y \implies f(x) \leq f(y).$$

2. f is decreasing if and only if

$$\forall x, y \in E : x \leq y \implies f(y) \leq f(x).$$

3. f is monotone if and only if f is either increasing or decreasing.

4. f is strictly increasing if and only if

$$\forall x, y \in E : \begin{cases} x \leq y \\ \text{and} \\ x \neq y \end{cases} \implies \begin{cases} f(x) \leq f(y) \\ \text{and} \\ f(x) \neq f(y). \end{cases}$$

5. f is strictly decreasing if and only if

$$\forall x, y \in E : \begin{cases} x \leq y \\ \text{and} \\ x \neq y \end{cases} \implies \begin{cases} f(y) \leq f(x) \\ \text{and} \\ f(x) \neq f(y). \end{cases}$$

6. f is strictly monotone if and only if f is strictly increasing or strictly decreasing.

Example 34

1. The mapping

$$\begin{aligned} f : (\mathbb{N}^*, |) &\longrightarrow (\mathbb{N}^*, |) \\ x &\longmapsto f(x) = x^2 \end{aligned}$$

is strictly increasing with respect to the divisibility relation $|$.

2. For any set E , the mapping

$$\begin{aligned} g : (\mathcal{P}(E), \subset) &\longrightarrow (\mathcal{P}(E), \subset) \\ X &\longmapsto g(X) = C_E^X \end{aligned}$$

is strictly decreasing with respect to the inclusion relation $\subset$.