Chapter 03 : Differential equations

I- Ordinary differential equations

1. Definitions
2. First order differential equations
3. second order equations with constant coefficients

1. Ordinary differential equations

1) Definition: Every relation between real variable x, unknown continuous function y, and its Derivatives $y', y'', y^{(3)}, \dots, y^{(n)}$ is called an ordinary differential equation.

Any ordinary differential equation (ODE) is presented by one of the following

$$(E)\ :\ F\big(x, y,\ y', \dots, y^{(n)}\big) = 0 \quad \text{ou} \quad y^{(n)} = F\big(x, y, y', \dots, y^{(n-1)}\big)$$

We call the integer n in equation (E) the order of the equation.

Examples :

$(E_1) : y' = xy + 3$ EDO of order 1.

$(E_2) : y'' + x^2 y' = x$ EDO of second order.

$(E_3) : x(y')^2 + y + e^x = 0$ EDO of first order.

$(E_4) : y'' - (y')^3 = \cos(x)$ EDO of order 2.

2) Solution of the ordinary differential equation

We call solution of the equation (E) every function φ, n-times deffrientiable on $I \subseteq \mathbb{R}$, and $F\big(x, \varphi,\ \varphi', \dots, \varphi^{(n)}\big) = 0$.

We call φ the integral of the equation(E) on $I \subseteq \mathbb{R}$. And the graph of φ Is called the integral curve of (E).

Theorem:

Let the differential equation $(E)\ :\ F\big(x, y,\ y', \dots, y^{(n)}\big) = 0$

If φ_1, and φ_2 are two solutions of (E) on $I \subseteq \mathbb{R}$.

Then: For any $\alpha, \beta \in \mathbb{R}$, $\alpha\varphi_1 + \beta\varphi_2$ is solution of (E).

Example :

Let $(E) : y'' - 5y + 6y = 0$, $\varphi_1(x) = e^{2x}$, and $\varphi_2(x) = e^{3x}$are solutions of (E).

Then: $\varphi(x) = \varphi_1(x) + \varphi_2(x) = e^{2x} + e^{3x}$ is solution of(E).

$(\varphi'(x) = 2e^{2x} + 3e^{3x}$, $\varphi''(x) = 4e^{2x} + 9e^{3x}$, $\varphi''(x) - 5\varphi'(x) + 6\varphi(x) = 0$)

3) Cauchy problem :

The Cauchy problem is written by : (P) : $\begin{cases} y^{(n)} = F\big(x, y, y', \dots, y^{(n-1)}\big) \\ y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1} \end{cases}$

Example : Solve the problem (P) : $\begin{cases} y' + 2xy = 0 \\ y(0) = 2 \end{cases}$

2. Ordinary differential equation of first order

They have the form : $f(x, y, y') = 0$ ou $y' = f(x, y)$

There are three main types of differential equations of first order.

- Differential equation with separate variables.
- Homogeneous differential equations
- Linear differential equations

And a finite number of special equations: Bernoulli equation, equation of Riccati, equation Of Lagrange, and Clairaut equation….

Resolution of differential equations of order 1 :

1) differential equations with separate variables: They have the form: $f(y)y' = g(x)$

$$\int f(y)dy = \int g(x)dx$$ (Find y as a function of x)

Examples : 1) (E_1) : $(1 + x^2)y' = xy$ EDO with separate variables

2) (E_2) : $x\,y' = y + xy$

3) (E_3) : $\begin{cases} y' = 2x\sqrt{y-1} \\ y(1) = 1 \end{cases}$

2) Homogeneous differential equations : They presented in the form : $y' = f\left(\frac{y}{x}\right)$

To solve this equation put the change of variable $z = \frac{y}{x}$

$z = \frac{y}{x} \Rightarrow y = xz$ et $y' = z + xz'$.

$y' = f\left(\frac{y}{x}\right) \Rightarrow z + xz' = f(z) \Rightarrow \frac{1}{f(z)-z} z' = \frac{1}{x}$ Equation with separate. variables.

Examples :

1) (E_1) : $(x^2 + y^2) - xyy' = 0$. Divide by x^2 .

2) $(E_2)\ :\ xy' = y + x\cos\left(\frac{y}{x}\right)$

3) $(E_3)\ :\ \begin{cases} x^2y' - 2xy + y^2 = 0 \\ y(1) = 2 \end{cases}$

3) Linear differential equations: $(E):\ y' + a(x)y = b(x)$,

a, b Two continuous functions on $I \subseteq \mathbb{R}$.

The lenear equation solved by two steps

1st Step : Find y_0 Solution of the equation (E_0) without the second member $(b(x) = 0)$.

$$(E_0)\ :\ y' + a(x)y = 0 \ \Rightarrow\ \frac{y'}{y} = -a(x) \ \Rightarrow\ y_0 = k\, e^{-\int a(x)dx}$$

2nd Step : Find y_p the particular solution of (E).

Use the method of constant variation. $k = k(x)$ (Function)

We have : $y = ke^{-\int a(x)dx} \Rightarrow y' = k' e^{-\int a(x)dx} - k\, a(x)e^{-\int a(x)dx}$

Replace in (E) : $k' e^{-\int a(x)dx} = b(x) \Rightarrow k = \int b(x)e^{\int a(x)dx}dx.$

Then : $y_p = ke^{-\int a(x)dx} = e^{-\int a(x)dx} \int b(x)e^{\int a(x)dx}dx.$

The general solution of (E) is given by : $y_G = y_0 + y_p.$

$y_G = k\, e^{-\int a(x)dx} + e^{-\int a(x)dx} \int b(x)e^{\int a(x)dx}dx.$

Examples :

1) $(E_1) : xy' + 2y = \frac{1}{2}x^3$

2) $(E_2) : \begin{cases} y' + 2y = e^x \\ y(0) = 1 \end{cases}$ $\quad a(x) = 2,\ b(x) = e^x$

3) $(E_3) : y' - \frac{y}{x} = x \arctan(x)$ $\quad a(x) = -\frac{1}{x}\ ,\ b(x) = x \arctan(x)$

4) Differential equation of Bernoulli :

They take the form: $(E)\ :\ y' + a(x)y = b(x)y^n,\ n > 1$ (integer)

To solve the Bernoulli equation put the change of variable: $z = \frac{1}{y^{n-1}}$.

$z = \frac{1}{y^{n-1}} \Rightarrow z' = \frac{1-n}{y^n}y'$ Replace in (E)

$(E): \frac{1}{y^n}y' + \frac{a(x)}{y^{n-1}} = b(x) \Rightarrow \frac{1}{1-n}z' + a(x)z = b(x)$ is linear ODE.

Examples :

1) $(E_1) : xy' + y = y^2 \ln(x)$. EDO of Bernoulli with $n = 2$.

2) $(E_2) : x^2y' + xy = y^5$ EDO of Bernoulli with $n = 5$.

5) Differential equation of Riccati :

They have the form : $(E) : y' + a(x)y^2 + b(x)y = c(x)$ with : $a, b,$ and c three continuous functions on the interval $I \subseteq \mathbb{R}$.

To solve the Riccati equation, we need to know a particular solution y_1 , and put $y = y_1 + \frac{1}{z}$

$y = y_1 + \frac{1}{z}$, et $y' = {y'}_1 - \frac{1}{z^2}z'$ Replace in(E), we get the following linear equation

$$z' - (2a(x)y_1 + b(x))z = a(x).$$

Examples :

1) $(E) : \begin{cases} x^2y' = xy - x^2y^2 - 1 \\ y(1) = 2 \end{cases}$ with $y_1 = \frac{1}{x}$ is a particular solution.

2) $(E) : y' = -2x + \left(1 + \frac{1}{x}\right)y + \frac{1}{x}y^2$, $y_0 = x$ The particular solution.

3) Différential equations of second order :

The différential equations of order 2 have the form : $y'' = F(x, y, y')$ or $F(x, y, y', y'') = 0$

There is two principal types of differential equations of order 2 :

Incomplete differential equations And Linear differential equations

1) Incomplete differential equations : Generally, There are three cases :

First case : $(E) : y'' = f(x)$

To solve it integrate two times.

$y'' = f(x) \Rightarrow y' = \int f(x)dx = F(x) + c$ $\quad F$ primitive de f.

$\Rightarrow y = \int (F(x) + c)dx = G(x) + cx + k$, G primitive of F.

Example : $(E) : y'' = \frac{1}{1+x}$

second case : $(E) : y'' = f(x, y')$

To solve it put the change : $y' = z$

$y' = z \Rightarrow z' = f(x, z)$ EDO of order one.

Example : $(E) \quad xy'' - y' = 0$

Third case : $(E) : y'' = f(y, y')$

Example : $(E) : y'' = \frac{y'^2}{\tan(y)}$

2) Linear differential equations : $(E): a(x)y'' + b(x)y' + c(x)y = f(x)$
With $a, b, c,$ and f continuous functions on the

Remark : In this chapter we interested a equations with a constant coefficients $(a, b,$ et $\in \mathbb{R})$

3) Linear differential equations of second order with a constant coefficients :

$$(E): \; ay'' + by' + cy = f(x)$$

With a, b, c real numbers, $a \neq 0$, and f a continuous function on $I \subseteq \mathbb{R}$.

i) Differential equations of second order homogeneous: (without a second member)

$(E_0): \; ay'' + by' + cy = 0, \; a, b, c$ des réels , et $a \neq 0$.

To solve it consider the solution $y = e^{rx}$ with $r \in \mathbb{C}$.

$y = e^{rx}, \; y' = re^{rx}$, et $y'' = r^2e^{rx}$ Replace in the equation(E_0), we find

$ar^2 + br + c = 0$.

We call the polynomial $P(r) = ar^2 + br + c$, a characteristic polynomial associated with equation (E_0).

Find the general solution of the homogeneous equation (E_0) we distinguish three cases :

First case : $\Delta = b^2 - 4ac > 0$.

P have two real roots: $r_1 = \frac{-b+\sqrt{\Delta}}{2\,a}$, $r_2 = \frac{-b-\sqrt{\Delta}}{2\,a}$

And the general solution is given by : $y_0 = C_1e^{r_1x} + C_2e^{r_2x}$

Examples

1) $(E_0): \; y'' + y' - 2y = 0$

2) $(E_0): \; 3y'' - 5y' + 2y = 0$

Second case : $\Delta = b^2 - 4ac = 0$.

P have one real root : $r_1 = r_2 = r = \frac{-b}{2a}$

The general solution given by: $y_0 = (C_1 x + C_2)e^{rx}$.

Examples :

1) $(E_0): \ y'' + 2y' + y = 0$

2) $(E_0): \ 9y'' - 6y' + y = 0$

Third case : $\Delta = b^2 - 4ac < 0$.

P have two complex roots : $r_1 = \alpha + i\beta$, $r_2 = \overline{r_1} = \alpha - i\beta$

The general solution given by $y_0 = \left(C_1 cos(\beta x) + C_2 sin(\beta x)\right)e^{\alpha x}$

Examples :

1) $(E_0): \ y'' - 2y' + 2y = 0$

2) $(E_0): \ y'' + 2y = 0$

ii) Differential equation of second order with a second member:

$(E): \ ay'' + by' + cy = f(x)$ with a, b, c a real , and $a \neq 0$.

The equation (E) solved in two steps :

First step: Find y_0 Solution of homogeneous equation $(E_0): ay'' + by' + cy = 0$.

Second step: Find y_p particular solution of (E).

$y_p = C_1 y_1 + C_2 y_2$ with y_1 and y_2 are solutions of the homogeneous equation (E_0).

Determine C_1 et C_2 By the constant variation method.

Integrate C'_1 and C'_2 Solution of system : $\begin{cases} C'_1 y_1 + C'_2 y_2 = 0 \\ C'_1 y'_1 + C'_2 y'_2 = \frac{1}{a} f(x) \end{cases}$

Conclusion : $y_G = y_0 + y_p$ general solution of (E).

Examples :

1) $(E) : 2y'' - 3y' + y = e^x$

2) $(E) : \ y'' + y = \sin(x)$

3) $(E):\begin{cases}4y''-4y'+y=xe^{\frac{1}{2}x}\\ y(0)=2\ ,\ \ y'(0)=1\end{cases}$

Remark : (Find y_p by substitution)

1) If the second member $f(x)=Q(x)e^{sx}$ with: Q polynômial and $s\in\mathbb{R}$.

we distinguish three cases :

$i)$ If s is not root of $P(r)$. Then : $y_p=R(x)e^{sx}$, R polynômial and $d°R=d°Q$

$ii)$ If s is simple root of $P(r)$. Then : $y_p=R(x)e^{sx}$ with $d°R=d°Q+1$

Or $\quad y_p = x\, R(x) e^{sx}$ with $d°R = d°Q$

$iii)$ If s is double root of $P(r)$. Then: $y_p = R(x)e^{sx}$ with $d°R = d°Q + 2$

Or $\quad y_p = x^2\, R(x)e^{sx}$ avec $d°R = d°Q$

2) If the second member $f(x) = A\, cos(\beta x) + B\, sin(\beta x)$
   we distinguish two cases :

$i)$ If βi is not root of $P(r)$. Then : $y_p = A_1 \cos(\beta x) + B_1 \sin(\beta x)$

$ii)$ If βi is a root of $P(r)$. Then : $y_p = x(A_1 \cos(\beta x) + B_1 \sin(\beta x))$.

Examples :

1) $(E) : y'' - 3y' + 2y = (2x^2 - 3)e^{-x}$

2) $(E) : y'' + 4y = 3\sin(2x)$