CHAPTER 2

IMPROPER INTEGRALS

The function that generate the Riemann integrals of chapter1 are continous on closed intervals. Thus, the functions are bounded and intervals are finite. Integrals of functions with these characteristics are called proper integrals. When one or more of these restrictions is relaxed, the integrals are said to be improper.
Categories of improper integrals are established below.
The integral $\int_a^b f(x)\,dx$ is called an improper integral if

1. $a = \infty$ or $b = \infty$ or both, i.e, one or both integration limitis is infinite,
2. $f(x)$ is unbounded at one or more points of $a \leq x \leq b$ such points are called singularities of $f(x)$.

Integrals correspoding to (1) and (2) are called improper integrals of the first and second kinds, respectively. Integrals with both conditions (1) and (2) are called improper integrals of the third kind.

Example 1 *$\int_0^\infty sinx^2 dx$ is an improper integral of the first kind.*

Example 2 *$\int_0^5 \frac{x}{x-2} dx$ is an improper integral of the second kind.*

Example 3 *$\int_0^\infty \frac{e^x}{x} dx$ is an improper integral of the third kind.*

2.1 Improper integrals of the first kind (unbounded intervals)

Definition 2.1 *Integrals with infinite limits of integration are called improper integrals of the first kind.*

1. *If f is integrable on $[a,\infty[$, then*

$$\int_a^{\infty} f(x)\,dx = \lim_{x\longrightarrow\infty}\int_a^x f(t)\,dt$$

.

2. *If f is integrable on $]-\infty,b]$, then*

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{x\longrightarrow-\infty}\int_x^b f(t)\,dt$$

.

3. *If f is integrable on $]-\infty,\infty[$, then*

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx\\ &= \lim_{x\longrightarrow-\infty}\int_x^c f(t)\,dt + \lim_{x\longrightarrow\infty}\int_c^x f(t)\,dt.\end{aligned}$$

where c is any real number.

In each case,if the limit exists and is finite, the improper integral is said to be convergent, if otherwise, it is called divergent.

Example 4 *Determine if the following integral is convergent or divergent*

$$\int_0^{+\infty} e^{-t}dt.$$

Example 5 *Investigate the convergece of*

$$\int_0^{+\infty} sintdt$$

Example 6 *(Riemann integral)*

$$\int_1^{+\infty} \frac{dt}{t^\alpha}$$

2.2 Improper integrals of the second kind

Definition 2.2 *Integrals of functions that become infinite at a point within the interval of integration are called improper integrals of thee second kind.*

1. *If $f(x)$ becomes unbounded only at the point a of the interval $[a,b]$, then*

$$\int_a^b f(x)\,dx = \lim_{x\longrightarrow a^+} \int_x^b f(t)\,dt.$$

2. *If $f(x)$ becomes unbounded only at the point b of the interval $[a,b]$, then*

$$\int_a^b f(x)\,dx = \lim_{x\longrightarrow b^-} \int_a^x f(t)\,dt.$$

3. *If $f(x)$ becomes unbounded only at an interior point c, of the interval $[a,b]$, then*

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x) \\ &= \lim_{x\longrightarrow c^-} \int_a^x f(t)\,dt + \lim_{x\longrightarrow c^+} \int_x^b f(t)\,dt.\end{aligned}$$

In each case, if the limit is finite, the improper integral converges and that limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

Example 7 *Determine if the integral converges or diverges. If the integral converges determine its value.*

$$\int_0^1 \frac{dx}{\sqrt{x}}.$$

Example 8 *Investigate the convergence of*

$$\int_2^5 \frac{dx}{x-5}$$

Example 9 *(Riemann integral) For what value of α does the integral*

$$\int_0^1 \frac{dx}{x^\alpha} dx$$

converges?When the integral does converge, What is its value?

Convergence test

Remark 2.1 *Convergence tests determine whether an improper integral converges or diverges.*

Theorem 2.2.1 *(Comparison test)*
Let f and g be two positive and integrable functions on $]a,b[$ where $b=\infty$ or $f(b)=\infty$ such that $0 \leq f(x) \leq g(x)$ for every $x \in]a,b[$, then

$$0 \leq \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

The inequalities above imply the following statements:

a) $\int_a^b g(x)\,dx$ *converges* $\Rightarrow \int_a^b f(x)\,dx$ *converges;*

b) $\int_a^b f(x)\,dx$ *diverges* $\Rightarrow \int_a^b g(x)\,dx$ *diverges.*

Example 10 *Determine wether* $I = \int_1^\infty e^{-x^2} dx$ *converges or diverges.*

Theorem 2.2.2 *(Quotient test)*
Let f and g be two positive and integrable functions on $]a,b[$ where $b=\infty$ or $f(b)=\infty$ such that

$$\lim_{x \longrightarrow +\infty} \frac{f(x)}{g(x)} = L$$

1. *If $L \neq 0$ and $L \neq \infty$, then the integrals $\int_a^b f(x)\,dx$, $\int_a^b g(x)\,dx$ both converge or both diverge.*
2. *If $L = 0$ and $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ converges.*
3. *If $L = \infty$ and $\int_a^b g(x)\,dx$ diverges, then $\int_a^b f(x)\,dx$ diverges.*

Example 11 *Determine wether* $I = \int_1^\infty \frac{dx}{\sqrt{x^6+1}}$ *converges or diverges.*

Example 12 *Determine wether* $I=\int_2^{\infty}\frac{x^2-x-1}{x^3+x^{\frac{1}{3}}}dx$ *converges or diverges.*

Definition 2.3 *Let f be an integrable function on $[a,b[$. We say that the improper integral $\int_a^b f(x)\,dx$ is absolutely convergent if $\int_a^b |f(x)|dx$ converge.*

Theorem 2.2.3 *Every absolutely convergent integral is convergent.*

Example 13 *Let* $I=\int_1^{\infty}\frac{sinx}{x^2}dx$ *be an improper integral of the first kind.*

Definition 2.4 *Let f be an integrable function on $[a,b[$. If $\int_a^b f(x)\,dx$ converges but $\int_a^b |f(x)|dx$ diverges, then $\int_a^b f(x)\,dx$ is called conditionally convergent.*

Example 14 $I=\int_1^{+\infty}\frac{sint}{t}dt$ *is conditionally convergent,*

Theorem 2.2.4 *(Abel's test)*
Let $f, g : [a, b[\longrightarrow \mathbb{R}$ *ba two integrable functions where*

1. f *is positive, decreasing and its limit is zero at* b *(*b *is singularity point or infini).*
2. $\exists M > 0$ *such that* $\forall x \in [a, b[$
$$\left| \int_a^x g(t)\, dt \right| \leq M.$$

Then $\int_a^b f(t)\, g(t)\, dt$ *converges.*

Example 15

$$I = \int_1^{+\infty} \frac{cost}{t^\alpha} dt,\ \alpha > 0.$$