Solution of Series of Exercises 2

Solution of Exercise 1

Let us study the simple convergence and uniform convergence of the following sequences of functions:

1. $f_n(x) = \dfrac{1-nx^2}{1+nx^2}$, $E_1 = \mathbb{R}$ then on $E_2 = [a, +\infty[$ $(a > 0)$.

1.1 **Simple Convergence :**

If $x = 0$, $f_n(0) = 1$ and if $x \neq 0$, $\lim f_n(x) = -1$. the sequence of fonctions (f_n) is therefore simply convergente on $E_1 = [-a, a]$ towards f, where $f(x) = \begin{cases} 1, & \text{if } x = 0, \\ -1, & \text{if } x \neq 0 \end{cases}$.

1.2 **Uniform Convergence** :

1.2.1. On $E_1 = \mathbb{R}$. The function f is not continuous on $\mathbb{R}$, it folloxs that (f_n) doesn't converge uniformly on E_1.

1.2.2. On $E_2 = [a, +\infty[$ $(a > 0)$. We have

$$\begin{aligned} \left|f_n(x) - f(x)\right| &= \left|\frac{1-nx^2}{1+nx^2} - (-1)\right| \\ &= \left|\frac{2}{1+nx^2}\right| \\ &\leq \frac{2}{1+na^2}. \end{aligned}$$

This later quantity tends towards 0, and the sequence of functions converges uniformly to -1 on $E_2 = [a, +\infty[$ $(a > 0)$.

2. $\mathrm{f}_n(x) = \dfrac{x}{1+nx}$, $E_3 = [0, 1]$.

2.1 **Simple Convergence :**

If $x = 0$, $f_n(0) = 0$ and if $x \neq 0$, $\lim f_n(x) = 0$.

The sequence of functions (f_n) is therefore simply convergent on $E_3 = [0, 1]$ to the zero function $f = 0$.

2.2 **Uniform Convergence :**

Since each function $x \mapsto f_n(x) - f(x) = \dfrac{x}{1+nx}$, is increasing on E_3, we then have $\sup\left|f_n(x) - f(x)\right| = \dfrac{1}{1+n}$. This later quantity tends to 0, and the sequence of functions converges uniformly to 0 on E_3

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3. $f_n(x) = \cos(\frac{5+nx}{n})$, $E_4 = \mathbb{R}$.

3.1 **Simple Convergence :**

If $x = 0$, $f_n(0) = \cos(\frac{5}{n})$, and $\lim f_n(0) = 1$.

When $x \neq 0$, $\lim f_n(x) = \cos(x)$.

In all cases, the sequence of functions (f_n) is therefore simply convergent on $E_4 = \mathbb{R}$ to f, where $f(x) = \cos(x)$.

3.2 **Uniform Convergence :**

We have

$$\begin{aligned}
\left|f_n(x) - f(x)\right| &= \left|\cos(\frac{5+nx}{n}) - \cos(x)\right| \\
&= \left|\cos(x) \times \cos(\frac{5}{n}) - \sin(x) \times \sin(\frac{5}{n}) - \cos(x)\right| \\
&\leq |\cos(x)| \times \left(\left|\cos(\frac{5}{n}) - 1\right| + |\sin(x)| \times \left|\sin(\frac{5}{n})\right|\right) \\
&\leq \frac{25}{2n^2} + \frac{5}{n}.
\end{aligned}$$

This later quantity tends to 0, and the sequence of functions converges uniformly to 0 on E_4.

4. $f_n(x) = \frac{\sin(nx)}{nx}$ and $f_n(0) = 0$, $E_5 = \mathbb{R}$ then on $E_6 = [a, +\infty[$ $(a > 0)$

4.1 **Simple Convergence :**

If $x = 0$, $f_n(0) = 0 \Rightarrow \lim f_n(0) = 0$, and if $x \neq 0$, $\lim f_n(x) = 0$. the sequence of functions (f_n) is therefore simply convergent on $\mathbb{R}$ to $f(x) = 0$.

4.2 **Uniform Convergence :**

4.2.1 On $E_5 = \mathbb{R}$. Let's consider the sequence of points $(x_n = \frac{\pi}{2n})$. We have $\left|f_n(x_n) - f(x_n)\right| = \frac{2}{\pi} \nrightarrow 0$. It follows that (f_n) doesn't converge uniformly on $E_5 = \mathbb{R}$.

4.2.2. On $E_6 = [a, +\infty[$ $(a > 0)$. We have

$$\left|f_n(x) - f(x)\right| = \left|\frac{\sin(nx)}{nx}\right| \leq \frac{1}{nx} \leq \frac{1}{na}.$$

This later quantity tends to 0, and the sequence of funtions converges uniformly to 0 on $E_6 = [a, +\infty[$ $(a > 0)$.

Solution of Exercise 1

5. $f_n(x) = x^n(1-x)$, $E_7 = [0,1]$.

5.1 **Simple Convergence :**

If $x = 0$ or $x = 1$, $f_n(0) = f_n(1) = 0$, and the sequence converges to 0. If $x \in]0,1[$, $\lim_{n\to+\infty} f_n(x) = 0$.

Finally, the sequence (f_n) converges simply on $[0,1]$ and this limit is the zero function $f(x) = 0$.

5.2 **Uniform Convergence :**

By performing a simple calculation, we find

$$\sup_{x\in[0,1]} \left|f_n(x) - f(x)\right| = a_n = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n.$$

This later quantity equivalent in neighborhood of infinity by $\frac{1}{ne}$.

Since this quantiry tends to 0, when n tends to $+\infty$, therefore the considered sequence converges uniformly to 0 on $[0,1]$.

Solution of Exercise 2

Let us consider the series of functions:

$$f_n(x) = \sin^2(x)\cos^n(x), \text{ for } n \geq 1 \text{ and } x \in \left[0, \frac{\pi}{2}\right].$$

1. If $x = 0$, $\sum_{n\geq 0} f_n(x) = 0$ and the series converges.

If $x \in \left]0, \frac{\pi}{2}\right]$, the series $\sum_{n\geq 0} f_n$, is geometric with ratio $q = \cos(x) \in [0,1[$, and therefore it converges on $\in \left]0, \frac{\pi}{2}\right]$.

Finally:

$$S(x) = \begin{cases} \dfrac{\sin^2(x)}{1-\cos(x)}, \text{ if } x \in \left]0, \dfrac{\pi}{2}\right] \\ 0, \text{ if } x = 0. \end{cases} \quad (1)$$

2. Since S is not continuous on the segment $\left[0, \frac{\pi}{2}\right]$, the series $\sum_{n\geq 0} f_n$, is therefore not uniformly convergent on this segment..

Solution of Exercise 3

Let us consider the series of functions:

$$f_n(x) = \frac{x}{(1+x^2)^n}, \text{ for } n \geq 1 \text{ and } x \in \mathbb{R},$$

1. If $x = 0$, $\sum_{n\geq 1} f_n(x) = 0$, the series is simply convergent.
If $x \neq 0$,, the series $\sum_{n\geq 1} f_n(x)$ is geometric series with ratio $q = \dfrac{1}{1+x^2} < 1$, and therefore it converges for all $x \neq 0$.
Finally :

$$S(x) = \begin{cases} \frac{1}{x}, \text{ if } x \neq 0 \\ 0, \text{ if } x = 0. \end{cases} \tag{2}$$

2. Since S is not continuous at the point 0, the series $\sum_{n\geq 1} f_n$, is therefore not uniformly convergent on $\mathbb{R}$.
3. Let's stydy the normal convergence on $[a, b]$ $(0 < a < b)$. We have

$$\begin{aligned} \left|f_n(x)\right| &= \frac{x}{(1+x^2)^n} \\ &\leq \frac{x}{(1+a^2)^n} \\ &\leq \frac{b}{(1+a^2)^n}, \forall x \in [a, b], \end{aligned}$$

which is the general term of convergent geometric series, and so $\sum_{n\geq 1} f_n$ converges uniformly on $[a, b]$.
4. Calculate $\sum_{n\geq 1} \int_1^e f_n(x)dx$:
Since the series is unifomly convergent on $[1, e]$, it is therefore integrable term by term on this segment. We then have

$$\begin{aligned} \sum_{n\geq 1} \int_1^e f_n(x)dx &= \int_1^e \sum_{n\geq 1} f_n(x)dx \\ &= \int_1^e \frac{dx}{x} \\ &= 1. \end{aligned}$$