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Sequences and Series of Functions

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Chapter 1

Sequences and series of functions

1.1 Sequences of functions

Let $\mathbb{k}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$ and let E and F be two non-empty subsets of $\mathbb{k}$.

Definition 1.1.1. We call a sequence of functions any application $f_n: \mathbb{N} \rightarrow \mathcal{E}$, where $\mathcal{E} = \mathcal{E}(E, F)$ is the set of applications of E in F .

1.1.1 Simple convergence

In general, to study the simple convergence of a sequence of functions $f_n(x)$ on a subset E of $\mathbb{R}$, we will try to fix the real x and we will study the corresponding numerical sequence.

Definition 1.1.2. A sequence of functions $(f_n)_{n \in \mathbb{N}}$ is simply convergent on E to a function $f(x_0)$, when the numerical sequences $(f_n(x_0))_{n \in \mathbb{N}}$ is convergent, for all

$x_0 \in E$. We thus define a function f on the domain E by:

$$f(x) = \lim_{n \rightarrow +\infty} f_n(x). \quad (1.1)$$

In this case, the function f is called the simple limit of the sequence of functions $(f_n)_{n \in \mathbb{N}}$.

Example 1.1.1. Consider the sequence of functions f_n defined on $\mathbb{R}$ by

$$f_n(x) = \frac{x}{x^2 + n}, \quad n \in \mathbb{N}. \quad (1.2)$$

If $x = 0$, $f_n(0) = 0$, and the sequence converges to 0, and if $x \neq 0$, $\lim_{n \rightarrow +\infty} f_n(x) = \lim_{n \rightarrow +\infty} \frac{x}{x^2 + n} = 0$.

Finally, the sequence f_n converges simply on $\mathbb{R}$, and its limit is $f(x) = 0$.

1.1.2 Uniform convergence

Definition 1.1.3. We say that the sequence of functions $(f_n)_{n \in \mathbb{N}}$ converges uniformly on E to a function f if and only if:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0 \text{ and } \forall x \in E, \text{ we have } |f_n(x) - f(x)| < \epsilon. \quad (1.3)$$

Remak 1.1. This integer n_0 obviously depends only on ϵ and not on x . While, if n_0 depends on both x and ϵ , we will say that the convergence is simple on E .

Remak 1.2. Uniform convergence on E implies simple convergence on E .

1.1.3 A sufficient condition for uniform convergence

Proposition 1.1.1. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions which converges simply on E to a function f . If there exists a positive sequence (b_n) that converges to 0, such that

$$\forall x \in E, |f_n(x) - f(x)| < b_n, \quad (1.4)$$

then the sequence of functions $(f_n)_{n \in \mathbb{N}}$ is uniformly convergent on E .

Proof. Suppose that b_n tends to 0, when $n \rightarrow +\infty$, that is:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \text{ such that } \forall n \geq n_0 \text{ we have } b_n < \frac{\epsilon}{2}. \quad (1.5)$$

Since:

$$|f_n(x) - f(x)| < b_n < \frac{\epsilon}{2}, \text{ for all } x \in E. \quad (1.6)$$

We find

$$\sup_{x \in E} |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon, \text{ for all } x \in E. \quad (1.7)$$

□

1.1.4 A necessary and sufficient condition for uniform convergence

Proposition 1.1.2. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions that converges simply on E to a function f . For $(f_n)_{n \in \mathbb{N}}$ to be uniformly convergent to f on E , it is necessary and sufficient that the numerical sequence (a_n) which is defined by:*

$$a_n = \sup_{x \in E} |f_n(x) - f(x)|, \quad (1.8)$$

is convergent to 0.

Proof. $\Rightarrow$ Suppose that $(f_n)_{n \in \mathbb{N}}$ be uniformly convergent to f , then:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0 \text{ and } \forall x \in E, |f_n(x) - f(x)| < \frac{\epsilon}{2}. \quad (1.9)$$

As a result:

$$a_n = \sup_{x \in E} |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon. \quad (1.10)$$

$\Leftarrow$ Now, let's assume that a_n tends to 0, when $n \rightarrow +\infty$, we then have:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, a_n < \epsilon. \quad (1.11)$$

As a result:

$$|f_n(x) - f(x)| \leq \sup_{x \in E} |f_n(x) - f(x)| = a_n < \epsilon, \text{ for all } x \in E. \quad (1.12)$$

□

Remak 1.3. A sufficient condition for the sequence of functions $(f_n)_{n \in \mathbb{N}}$ doesn't converge uniformly to f on E is the existence of a sequence of points $(x_n) \subset E$, verifying:

$$|f_n(x_n) - f(x_n)| \rightarrow 0, \text{ when } n \rightarrow +\infty. \quad (1.13)$$

Example 1.1.2. Let the sequence of functions f_n be defined on $[0, 1]$ by:

$$f_n(x) = x^n(1 - x), \quad n \in \mathbb{N}. \quad (1.14)$$

* If $x = 0$ or $x = 1$, $f_n(0) = f_n(1) = 0$, and the sequence converges to 0.

* If $x \in]0, 1[$, $\lim_{n \rightarrow +\infty} f_n(x) = 0$.

Finally, the sequence f_n simply converges on $[0, 1]$ and its limit is the zero function $f(x) = 0$.

By performing a simple calculation, we find

$$\sup_{x \in [0, 1]} |f_n(x) - f(x)| = a_n, \text{ such that } a_n = \frac{1}{n+1} \left(\frac{n}{n+1} \right)^n. \quad (1.15)$$

This last quantity is equivalent to the neighborhood of infinity by $\frac{1}{ne}$. Since this quantity tends to 0, when n tends to $+\infty$, the sequence of functions considered converges uniformly to 0 on the segment $[0, 1]$.

1.1.5 Cauchy criterion for uniform convergence

Proposition 1.1.3. For the sequence of functions $(f_n)_{n \in \mathbb{N}}$ to be uniformly convergent to f on E , it is necessary and sufficient that:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N}, p > q \geq n_0 \text{ and } \forall x \in E, |f_p(x) - f_q(x)| < \epsilon. \quad (1.16)$$

Proof. $\Rightarrow$ Suppose that $(f_n)_{n \in \mathbb{N}}$ be uniformly convergent to f , then:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0 \text{ and } \forall x \in E, |f_n(x) - f(x)| < \frac{\epsilon}{2}. \quad (1.17)$$

Let $\epsilon > 0$, then for all $p > q \geq n_0$, we have:

$$\begin{aligned} |f_p(x) - f_q(x)| &\leq |f_p(x) - f(x)| + |f_q(x) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ for all } x \in E. \end{aligned}$$

⊆ Now let's assume that:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N}, p > q \geq n_0 \text{ and } \forall x \in E, |f_p(x) - f_q(x)| < \epsilon. \quad (1.18)$$

Let p tend towards $+\infty$, we find the result. $\square$

1.1.6 Properties of sequences of functions

Continuity

Proposition 1.1.4. *Let (f_n) be a sequence of continuous functions on a segment $[a, b]$, converging uniformly on the same segment to a function f . Then f is a continuous function on $[a, b]$.*

Proof. Let x_0 be any point of $[a, b]$.

f_n is continuous at the point x_0 , then:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in [a, b], |x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}. \quad (1.19)$$

$(f_n)_{n \in \mathbb{N}}$ converges uniformly to f , then:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0 \text{ and } \forall x \in E, |f_n(x) - f(x)| < \frac{\epsilon}{3}. \quad (1.20)$$

We can write:

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &= |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

$\square$

Remak 1.4. *Under the conditions of the previous proposition, we can write:*

$$\lim_{x \rightarrow x_0} \lim_{n \rightarrow +\infty} f_n(x) = \lim_{n \rightarrow +\infty} \lim_{x \rightarrow x_0} f_n(x) = f(x_0). \quad (1.21)$$

Integrability

Proposition 1.1.5. *Let (f_n) be a sequence of continuous functions on a segment $[a, b]$, converging uniformly on the same segment to a function f . Then f is an integrable function on $[a, b]$, and moreover:*

$$\int_a^b f(x)dx = \lim_{n \rightarrow +\infty} \int_a^b f_n(x)dx. \quad (1.22)$$

Proof. Under the assumptions of proposition 1.1.5, the uniform limit f is also continuous, which ensures the integrability of $f_n(x)$ and f .

$(f_n)_{n \in \mathbb{N}}$ converges uniformly to f , then:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0 \text{ and } \forall x \in E, |f_n(x) - f(x)| < \frac{\epsilon}{b-a}. \quad (1.23)$$

We can write:

$$\begin{aligned} \left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| &\leq \int_a^b |f_n(x) - f(x)| dx \\ &< \frac{\epsilon}{b-a} \left| \int_a^b dx \right| = \epsilon. \end{aligned}$$

□

Corollary 1.1.1. *Under the assumptions of proposition 1.1.5, we deduce that the sequence of integrals $\left(\int_a^x f_n(y)dy \right)_n$ is uniformly convergent to $\left(\int_a^x f(y)dy \right)_n$, for all $x \in [a, b]$.*

Proof. Since the integer n_0 in the relation (1.23) does not depend on b , it suffices to replace b by x . □

Differentiability

Proposition 1.1.6. *Let (f_n) be a sequence of functions defined on the segment $[a, b]$ and verify the following three conditions:*

1. $f_n, n = 0, 1, \dots$ are of class C^1 on a segment $[a, b]$.
 2. (f_n) simply converges on the same segment to a function f .
 3. The sequence of derivatives (f'_n) converges uniformly to a function g .
- Then, the sequence of functions (f_n) converges uniformly to a derivable function f and moreover $f' = g$.*

Proof. Since (f_n) is a sequence of continuous functions on a segment $[a, b]$, converging uniformly on the same segment to a function g , then the use of the proposition of integration affirms that f_n is an integrable function on $[a, b]$, and moreover:

$$f_n(x) = f_n(a) + \int_a^x f_n'(x) dx. \quad (1.24)$$

According to Corollary 1.1.1, the sequence $\left(\int_a^x f_n'(x) dx\right) f_n$ converges uniformly, and the numerical sequence $(f_n(a))$ is also convergent, $(f_n(x))$ is therefore the sum of two uniformly convergent sequences, so it is uniformly convergent.

We have:

$$\lim_{n \rightarrow +\infty} \int_a^x f_n'(x) dx = \int_a^x g(x) dx, \quad (1.25)$$

On the other hand:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_a^x f_n'(x) dx &= \lim_{n \rightarrow +\infty} (f_n(x) - f_n(a)) \\ &= f(x) - f(a). \end{aligned} \quad (1.26)$$

Using (1.25) and (1.26), we get:

$$\int_a^x g(x) dx = f(x) - f(a). \quad (1.27)$$

We derive this last equality, we find:

$$g(x) = f'(x). \quad (1.28)$$

□

Remak 1.5. Under the conditions of the previous proposition, we can write:

$$\lim_{n \rightarrow +\infty} \left(\frac{\partial}{\partial x} f_n(x) \right) = \frac{\partial}{\partial x} \left(\lim_{n \rightarrow +\infty} f_n(x) \right) = f'(x_0). \quad (1.29)$$

1.2 Series of functions

Definition 1.2.1. Let (f_n) be a sequence of functions from E to $\mathbb{k}$. A series of functions with general term f_n is any expression of the form $\sum_{n=0}^{+\infty} f_n(x)$.

Let $S_n(x) = \sum_{k=0}^n f_k(x)$, $n \in \mathbb{N}$ and $x \in E$.

S_n is called the partial sum of order n of the series $\sum_{n \geq 0} f_n(x)$.

1.2.1 Simple convergence

Definition 1.2.2. A series of functions with general term f_n is said to be simply convergent on a subset E of $\mathbb{R}$, if for all $x \in E$, the numerical series with general term $f_n(x)$ converges.

The term:

$$R_n(x) = S(x) - S_n(x) = \sum_{k=n+1}^{+\infty} f_k(x), \quad n \in \mathbb{N} \text{ and } x \in E. \quad (1.30)$$

is called the rest of order n of the series.

The convergence of the series of general term f_n is then expressed by the convergence of the sequence of partial sums $(S_n(x))$ to a function S That is to say:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0 \text{ and } \forall x \in E, |S_n(x) - S(x)| = |R_n(x)| < \epsilon. \quad (1.31)$$

Example 1.2.1. Consider the series of functions with a general term

$$f_n(x) = \frac{x^n}{\sqrt{n} + 1}, \quad n \geq 0 \text{ and } x \in \mathbb{R}. \quad (1.32)$$

For $x \neq 0$, the d'Alembert criterion gives us $\lim_{n \rightarrow +\infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| = |x|$. The series converges when $|x| < 1$ and diverges when $|x| > 1$.

If $x = -1$, the series becomes alternating and verifies the convergence criterion. If $x = 1$, it diverges.

Finally, the series of functions converges simply on $[-1, 1[$.

1.2.2 Uniform convergence

Definition 1.2.3. A series of functions with general term f_n , converges uniformly on a subset E of $\mathbb{R}$ and has the sum S , when the sequence of its partial sums is uniformly convergent on E , that is:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \sup_{x \in E} |S_n(x) - S(x)| = \sup_{x \in E} |R_n(x)| < \epsilon. \quad (1.33)$$

To say that the sequence of partial sums converges uniformly on E therefore means that $(R_n)_{n \in \mathbb{N}}$ converges uniformly to 0 on E .

Remak 1.6. We can define a norm of uniform convergence of S_n on E by:

$$\|S_n\| = \sup_{x \in E} |S_n(x)|. \quad (1.34)$$

The series of functions with a general term f_n converges uniformly and with a sum S if and only if the numerical sequence $(\|S_n - S\|)_{n \in \mathbb{N}}$ converges to 0.

1.2.3 Cauchy criterion for uniform convergence

Theorem 1.2.1. For the series of functions with general term f_n to be uniformly convergent on E , it is necessary and sufficient that:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N}, p > q \geq n_0 \text{ and } \forall x \in E, \sup_{x \in E} \left| \sum_{k=q+1}^p f_k(x) \right| < \epsilon. \quad (1.35)$$

Proof. The proof of this theorem is the same as for sequences by reasoning on the sequence of partial sums. $\square$

Corollary 1.2.1. The use of the uniform Cauchy criterion is often by its contraposition, to show that a series of functions does not converge uniformly.

1.2.4 A necessary condition for uniform convergence

Proposition 1.2.1. For a series of functions to be uniformly convergent, it is necessary that its general term tends to 0 uniformly.

Proof. It suffices to apply the uniform Cauchy criterion on:

$$\|f_n\| = \sup_{x \in E} \|f_n(x)\| = \sup_{x \in E} \|S_n(x) - S_{n-1}\|. \quad (1.36)$$

$\square$

1.2.5 A sufficient condition for uniform convergence (Weierstass criterion)

Proposition 1.2.2. (Proposition and definition) Let $\sum_{n \geq 0} f_n(x)$ be a series of functions defined on E . Suppose that there exists a positive numerical series $\sum_{n \geq 0} b_n$, such that:

$$\forall x \in E, |f_n(x)| < b_n. \quad (1.37)$$

If $\sum_{n \geq 0} b_n$ converges, then the series of functions $\sum_{n \geq 0} f_n(x)$ is absolutely and uniformly convergent on E .

In this case, we say that the series of functions $(f_n)_{n \in \mathbb{N}}$ is **normally** convergent on E .

Proof. From the inequality (1.37) and the comparison theorem, we deduce absolute convergence.

On the other hand, the numerical series $\sum_{n \geq 0} b_n$ converges, that is:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \text{ such that } \forall n \geq n_0, \sum_{k \geq n+1} b_k < \epsilon. \quad (1.38)$$

So

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \left| \sum_{k \geq n+1} f_k(x) \right| \leq \sum_{k \geq n+1} |f_k(x)| \leq \sum_{k \geq n+1} b_k < \epsilon. \quad (1.39)$$

This latter quantity independent of x , the rest of the series $\sum_{n \geq 0} f_n(x)$ converges uniformly to 0, the series $\sum_{n \geq 0} f_n(x)$ is therefore uniformly convergent. $\square$

Example 1.2.2. The series of functions $\sum_{n \geq 0} \frac{\sin(nx)}{\alpha^n}$, $\alpha > 1$ is normally convergent on $\mathbb{R}$, since $\left| \frac{\sin(nx)}{\alpha^n} \right| \leq \left(\frac{1}{\alpha} \right)^n$, general term of a convergent geometric series.

1.2.6 Necessary and sufficient condition for normal convergence

Proposition 1.2.3. *For the series of functions $\sum_{n \geq 0} f_n$ to be normally convergent on E , it is necessary and sufficient that the numerical series (a_n) with general term:*

$$a_n = \sup_{x \in E} |f_n(x)|, \quad (1.40)$$

be convergent.

Proof. $\Rightarrow$ When the series of functions $\sum_{n \geq 0} f_n$ is normally convergent on E , there exists a positive convergent series of term b_n verifying:

$$\forall x \in E, |f_n(x)| \leq b_n, \quad (1.41)$$

As result

$$a_n = \sup_{x \in E} |f_n(x)| \leq b_n, \quad (1.42)$$

and the series $\sum_{n \geq 0} a_n$ is convergent.

$\Leftarrow$ Now let's assume that $\sum_{n \geq 0} a_n$ is convergent, then we have:

$$|f_n(x)| \leq \sup_{x \in E} |f_n(x)| = a_n < \epsilon, \quad x \in E, \quad (1.43)$$

this is the definition of a normally convergent series. $\square$

Example 1.2.3. *The series of functions $\sum_{n \geq 0} f_n(x)$ defined on $[0, 1]$, such that:*

$$f_n(x) = \begin{cases} x^n \ln^2 x & \text{if } x \in]0, 1], \\ 0, & \text{if } x = 0 \end{cases} \quad (1.44)$$

We have:

$$\dot{f}_n(x) = \ln(x) (2 + n \ln(x)) x^{n-1} = 0, \quad \text{if } x = \exp\left(\frac{-2}{n}\right) = x_n. \quad (1.45)$$

As result

$$a_n = \sup_{x \in [0, 1]} |f_n(x)| = f_n(x_n) = \frac{4}{n^2 e^2}, \quad (1.46)$$

general term of a convergent series. The series of functions is normally convergent on $[0, 1]$.

Proposition 1.2.4. *The normal convergence of a series of functions on a subset E of $\mathbb{R}$ implies the uniform convergence of this series on E , and the converse is false.*

Proof. When the series of functions $\sum_{n \geq 0} f_n$ is normally convergent on E , the proof proceeds from the inequality:

$$\sup_{x \in E} \left| \sum_{k=q+1}^p f_k(x) \right| \leq \sum_{k=q+1}^p \sup_{x \in E} |f_k(x)| \quad (1.47)$$

and the Cauchy criterion.

The converse of this proposition is false. As an example, we take the series of functions with a general term

$$f_n(x) = \frac{(-1)^n}{n+x}, \quad x \in [0, 1] \text{ and } n \geq 1. \quad (1.48)$$

This series is uniformly convergent without being normally convergent on $[0, 1]$.

On the other hand:

$$|R_n(x)| \leq |f_{n+1}(x)| = \frac{1}{n+1+x} < \frac{1}{n+1} < \epsilon, \text{ for all } x \in [0, 1], \quad (1.49)$$

which shows uniform convergence on $[0, 1]$.

By against:

$$\sup_{x \in [0, 1]} |f_n(x)| = \frac{1}{n}, \quad (1.50)$$

general term of a divergent series, the series is therefore not normally convergent. $\square$

1.3 Properties of series of functions

1.3.1 Continuity

Proposition 1.3.1. *Let be a series of functions of general term f_n , defined on the interval $[a, b]$, which converges uniformly and of sum S on $[a, b]$. If f_n is continuous*

on $[a, b]$, for all $n \in \mathbb{N}$, then S is also continuous on $[a, b]$, and moreover, we have the following equality:

$$\lim_{x \rightarrow x_0} \sum_{n \geq 0} f_n(x) = \sum_{n \geq 0} \lim_{x \rightarrow x_0} f_n(x) = S(x_0), \text{ for all } x_0 \in [a, b], \quad (1.51)$$

which is a case of *inversion of limit and infinite sum*.

Proof. It suffices to apply Proposition 1.1.4 to the sequence (S_n) of partial sums of the series $\sum_{n \geq 0} f_n$, which are continuous as finite sums of continuous functions. $\square$

Remak 1.7. The condition of uniform convergence of the series of functions is sufficient but not necessary to ensure the continuity of the sums.

Remak 1.8. When the series of continuous functions of general term f_n simply converges on $[a, b]$ and has as sum a discontinuous function S , then $\sum_{n \geq 0} f_n$ does not converge uniformly on this interval.

Example 1.3.1. The series of general term continuous functions:

$$f_n(x) = \sin^2(x) \cos^n(x), \quad x \in \left[0, \frac{\pi}{2}\right], \quad n \in \mathbb{N}. \quad (1.52)$$

converges simply on $\left[0, \frac{\pi}{2}\right]$ and has the sum:

$$S(x) = \begin{cases} \frac{\sin^2(x)}{1 - \cos(x)}, & \text{if } x \in \left]0, \frac{\pi}{2}\right] \\ 0, & \text{if } x = 0. \end{cases} \quad (1.53)$$

Since S is discontinuous at 0, $\sum_{n \geq 0} f_n$ does not converge uniformly on $\left[0, \frac{\pi}{2}\right]$.

1.3.2 Integrability

Proposition 1.3.2. Let a series of functions with general term f_n , defined on $[a, b]$, converges uniformly and with sum S on $[a, b]$. If f_n is continuous on $[a, b]$, for all $n \in \mathbb{N}$, then, the series with general term $\int_a^b f_n(x) dx$ converges and has the sum $\int_a^b S(x) dx$, and moreover, we have the following equality:

$$\int_a^b S(x) dx = \sum_{n \geq 0} \int_a^b f_n(x) dx = \int_a^b \left(\sum_{n \geq 0} f_n(x) \right) dx,$$

which is a case of *intversion sum and integral*.

Proof. It suffices to apply Proposition 1.1.5 to the sequence of partial sums $(S_n)_{n \in \mathbb{N}}$ of the series $\sum_{n \geq 0} f_n(x)$. $\square$

Example 1.3.2. Let the series of functions with general term:

$$f_n(x) = \frac{x^{2n}}{(2n)!}, \quad x \in [0, 1].$$

This series converges uniformly on $[0, 1]$, since $|f_n(x)| \leq \frac{1}{(2n)!}$, for all $x \in [0, 1]$. According to the previous proposition, we then have:

$$\begin{aligned} \int_0^x \left(\sum_{n \geq 0} \frac{x^{2n}}{(2n)!} dx \right) &= \sum_{n \geq 0} \int_0^x \frac{x^{2n}}{(2n)!} dx \\ &= \sum_{n \geq 0} \frac{x^{2n+1}}{(2n+1)!} \\ &= \sinh(x), \text{ for all } x \in [0, 1]. \end{aligned}$$

1.3.3 Derivability

Proposition 1.3.3. Consider a series of functions with general term f_n , derivable on the segment $[a, b]$ and verifying:

1. The series of functions $\sum_{n \geq 0} f_n(x)$ converges simply on $[a, b]$.
2. The series of derivatives of general term f'_n converges uniformly on $[a, b]$ and has as sum a function g .

Then, the series of general term f_n is derivable term by term, and we have:

$$\dot{S}(x) = \frac{\partial}{\partial x} \left(\sum_{n \geq 0} f_n(x) \right) = \sum_{n \geq 0} \frac{\partial}{\partial x} f_n(x) = g(x),$$

Proof. It suffices to apply Proposition 1.1.5 to the sequence of partial sums $(S_n)_{n \in \mathbb{N}}$ of the series of general term f_n , which is derivable as finite sums of derivable functions. $\square$

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