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Analysis 3

Solutions of exercises about numerical series

Solution of exercise 1:

1. We have:

$$\begin{aligned} u_n &= \frac{1}{n(n-1)} \\ &= \frac{1}{n-1} - \frac{1}{n}, \end{aligned}$$

which implies :

$$\begin{aligned} \lim_{n\to+\infty} S_n &= \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\ &= \lim_{n\to+\infty} \left(1 - \frac{1}{n} \right) = 1. \end{aligned}$$

Hence $\sum_{n=2}^{+\infty} \frac{1}{n(n-1)}$ converges with sum $S = 1$.

2. Since:

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{n^2}{n!} &= \sum_{n=1}^{+\infty} \frac{n}{(n-1)!} \\ &= \sum_{n=1}^{+\infty} \left(\frac{n-1}{(n-1)!} + \frac{1}{(n-1)!} \right) \\ &= \sum_{n=2}^{+\infty} \frac{1}{(n-2)!} + \sum_{n=1}^{+\infty} \frac{1}{(n-1)!} \\ &= 2e, \end{aligned}$$

we then have $\sum\limits_{n=0}^{+\infty} \frac{n^2}{n!}$ converges with sum $S = 2e$.

3. Since

$$\begin{aligned}\sum_{n=0}^{+\infty} (-1)^n \frac{\cos(nx)}{2^n} &= Reel\left(\sum_{n=0}^{+\infty}\left(-\frac{\exp(ix)}{2}\right)^n\right)\\ &= Reel\left(\frac{2}{2+\exp(ix)}\right)\\ &= \frac{4+2\cos x}{5+4\cos x}.\end{aligned}$$

we then have $\sum\limits_{n=0}^{+\infty} (-1)^n \frac{\cos(nx)}{2^n}$ converges with sum $S = \frac{4+2\cos x}{5+4\cos x}$.

Solution of exercise 2:

Let us study the nature of the proposed numerical series

1. Since

$$\lim_{n\to+\infty} n\sin\left(\frac{1}{n}\right) = 1 \neq 0,$$

the series $\sum\limits_{n=1}^{+\infty} n\sin\left(\frac{1}{n}\right)$ is divergent .

2. We have

$$\arctan\left(\frac{1}{n^2}\right) \overset{V(+\infty)}{\sim} \frac{1}{n^2},$$

general term of convergent series (Riemann's series with $\alpha = 2 > 1$).

Therefore $\sum\limits_{n=1}^{+\infty} \arctan\left(\frac{1}{n^2}\right)$ converges.

3. $\sum\limits_{n=1}^{+\infty} \frac{\alpha^n}{\alpha^{2n}+\alpha^n+1}$ $(\alpha \geq 0)$. Let $u_n = \frac{\alpha^n}{\alpha^{2n}+\alpha^n+1}$.

3.1. If $\alpha = 0$, $\sum\limits_{n=1}^{+\infty} u_n = 0$, the proposed series converges.

3.2. If$\alpha = 1$, $u_n = \frac{1}{3} \nrightarrow 0$, when $n \to +\infty$, the series diverges.

3.2. If $\alpha \in]0,1[$. Consider the geometric series of general terlm $v_n = \alpha^n$, which is convergent.

Since $\lim_{n\to+\infty} \dfrac{u_n}{v_n} = 1$, the both series $\sum\limits_{n=1}^{+\infty} u_n$ and $\sum\limits_{n=1}^{+\infty} v_n$ are therfore the same nature, which shows the convergence of the series $\sum\limits_{n=1}^{+\infty} u_n$.

3.4. If $\alpha > 1$. Consider the geometric series of the general term $v_n = \left(\dfrac{1}{\alpha}\right)^n$ which is convergent.

Since $\lim_{n\to+\infty} \dfrac{u_n}{v_n} = 1$, the both series $\sum\limits_{n=1}^{+\infty} u_n$ and $\sum\limits_{n=1}^{+\infty} v_n$ are therfore the same nature, which shows the convergence of the series $\sum\limits_{n=1}^{+\infty} u_n$.

4. Using the Cauchy's creterion, we get

$$\begin{aligned}\lim_{n\to+\infty}\left(\left(\frac{n+a}{n+b}\right)^{n^2}\right)^{\frac{1}{n}} &= \lim_{n\to+\infty}\left(\frac{1+\frac{a}{n}}{1+\frac{a}{n}}\right)^n \\ &= \exp(a-b).\end{aligned}$$

Therefore:

4.1. If $a < b, \exp(a-b) < 1$, the series is convergent.

4.2. 4.1. If $a > b, \exp(a-b) > 1$, the series is divergent

4.3. If $a = b, u_n = 1 \nrightarrow 0$, when $n \to +\infty$, the proposed series is divergent.

5. We have $\dfrac{\ln(n)}{n^2+2} < \dfrac{\ln(n)}{n^2}$, general term of Bertrand's series which is convergent, the series $\sum\limits_{n=1}^{+\infty} \dfrac{\ln(n)}{n^2+2}$ is therefore convergent.

6. Let $u_n = \dfrac{1\times 3\times \ldots\times(2n-1)}{2\times 4\times\ldots\times(2n)}$, we then have:

$$\lim_{n\to+\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to+\infty}\frac{2n+1}{2n+2} = 1.$$

According to d'Alembert, we can't conclude. Usig the Raab

Duhamel creterion, we get:

$$\lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to+\infty}\frac{n}{2n+2}=\frac{1}{2}<1.$$

Consequently, the series $\sum\limits_{n=1}^{+\infty}\dfrac{1\times 3\times ...\times(2n-1)}{2\times 4\times ...\times(2n)}$ is divergent.

7. Let $u_n=\dfrac{2^n}{n^2}\sin^{2n}(\theta), \theta\in\left[0,\dfrac{\pi}{2}\right]$.

8. If $x=2k\pi$, $u_n=\dfrac{1}{n}$ general term of harmonic series which is divergent.

If $x\neq 2k\pi$. We can write

$$u_n=a_n\times b_n,$$

where $a_n=\exp(inx)$ and $b_n=\dfrac{1}{n}$.

We then have

$$\begin{aligned}\left|\sum_{k=1}^{n}a_k\right| &= \left|\sum_{k=1}^{n}\exp(ikx)\right|=\left|\exp(ix)\times\left(\frac{1-\exp(inx)}{1-\exp(ix)}\right)\right|\\ &= \left|\left(\frac{1-\exp(inx)}{1-\exp(ix)}\right)\right|\\ &\leq \left|\frac{2}{1-\exp(ix)}\right|\\ &= \frac{2}{|1-\cos x-i\sin x|}\\ &= \frac{2}{2\left|\sin\frac{x}{2}\right|\times\left|\sin\frac{x}{2}+i\cos\frac{x}{2}\right|}\\ &= \frac{1}{\left|\sin\frac{x}{2}\right|}.\end{aligned}$$

In the auther hand, the sequence (b_n) is decreasing and tends towards zero, therefore, according to Abel, the series $\sum\limits_{n=1}^{+\infty}\dfrac{\exp(inx)}{n}$

is convergent

Solution of exercise 3:

1. Consider the numerical series of general term

$$u_n = \frac{\sin n}{n^\theta}.$$

1. 1. Let's show that: if $\theta > 1$, the series is absolutely convergent.

Indeed: If $\theta > 1$, we have $|u_n| \leq \frac{1}{n^\theta}$, general term of convergent series, therefore $\sum\limits_{n=1}^{+\infty} \frac{\sin n}{n^\theta}$ is absolutely convergent.

1. 2. If v0 $< \theta \leq 1$, we have

$$\begin{aligned} |u_n| &= \left|\frac{\sin n}{n^\theta}\right| \geq \frac{\sin^2 n}{n^\theta} \\ &= \frac{1}{2n^\theta} - \frac{\cos(2n)}{2n^\theta} \\ &= v_n. \end{aligned}$$

Since $\sum\limits_{n=1}^{+\infty} \frac{1}{n^\theta}$ diverges and $\sum\limits_{n=1}^{+\infty} \frac{\cos(2n)}{n^\theta}$ converges (according to Abel), the series $\sum\limits_{n=1}^{+\infty} v_n$ is therfore convergent, it follows that $\sum\limits_{n=1}^{+\infty} u_n$ does not converge absolutely. But the series $\sum\limits_{n=1}^{+\infty} u_n$ converges (according to Abel), hence $\sum\limits_{n=1}^{+\infty} u_n$ is semi-convergent.

1. 3. Whereas, if $\theta \leq 0$, the series is divergent.

Indeed, if $\alpha \leq 0$, $\lim_{n\to+\infty} u_n$ does not exist, the series $\sum\limits_{n=1}^{+\infty} u_n$ is therefore divergent.

2.. Consider the series $\sum\limits_{n=1}^{+\infty} \left(\frac{2n-1}{n+1}\right)^{2n} \theta^n$, $\theta \in \mathbb{R}$.

Let $u_n = \left(\frac{2n-1}{n+1}\right)^{2n} \theta^n$. Using the Cauchy's creterion of the series absolutely convergente, we find:

$$\lim_{n\to+\infty} |u_n|^{\frac{1}{n}} = 4\,|\theta|\,.$$

Therefore:

2. 1. If $|\theta| < \frac{1}{4}$, the series is absolutely convergent.

2. 2. If $|\theta| > \frac{1}{4}$, the series is divergent.

2. 3. If $|\theta| = \frac{1}{4}$, the general term can be written as:

$$|u_n| = \left(\frac{2n-1}{2n+2}\right)^{2n} = \left(1 - \frac{3}{2n+2}\right)^{2n},$$

which tends to $\exp(-3) \neq 0$, when $n \to +\infty$, the series is therefore divergent.

3. Consider the series $\sum\limits_{n=1}^{+\infty} \frac{\cos(n)}{n^{\theta} + \cos(n)}$, $\theta \in \mathbb{R}$.

3.1. If $\theta \leq 0$, $\lim u_n$ does not exist, and the series diverges.

3.2. If $\theta > 1$, we have $|u_n| = \left|\frac{\cos(n)}{n^{\theta} + \cos(n)}\right| \leq \left|\frac{1}{n^{\theta}}\right|$. Since $\frac{1}{n^{\theta}}$ is general term of convergent series(Riemann's series with $\theta > 1$), then the series $\sum\limits_{n=1}^{+\infty} \frac{\cos(n)}{n^{\theta} + \cos(n)}$ is absolutely convergent.

3.3. If $0 < \theta \leq 1$. We can write $u_n = \frac{\cos(n)}{n^{\theta}} \times \frac{1}{1 + \frac{\cos(n)}{n^{\theta}}}$.

By using the Taylor expansion of the function $x \mapsto \frac{1}{1+x}$ in neighborhood of 0, we get:

$$\begin{aligned} u_n &= \frac{\cos(n)}{n^{\theta}}\left(1 - \frac{\cos(n)}{n^{\theta}} + \frac{\cos(n)}{n^{\theta}}\varepsilon(n)\right) \\ &= \frac{\cos(n)}{n^{\theta}} - \frac{\cos^2(n)}{n^{2\theta}} + \frac{\cos^2(n)}{n^{2\theta}}\varepsilon(n), \text{ where } \varepsilon(n) \to 0, \text{ when } n \to +\infty. \end{aligned} \tag{1}$$

We therefore have the sum of two absolutely convergent series and one convergent series. Therefore, the considered series is convergent.