Chapter 2: Free Linear Systems with One Degree of Freedom

(*Undamped Free Vibration*)

The most fundamental system for the study of vibrations is the single degree of freedom system. By definition, a single degree of freedom system is one for which only one independent coordinate is needed to completely describe the motion of the system.
It is called a *harmonic oscillator* when, as soon as it is displaced from its equilibrium position by a distance x (or angle θ), it is subjected to a restoring force that is opposite and proportional to the displacement x (or θ):

$\mathbf{F = -C \cdot x}$

To study vibratory systems, the following steps must be followed:

- Establish the differential equation that represents the motion.
- Solve the differential equation.
- Derive the physical parameters: Amplitude, Beat, Frequency, etc.

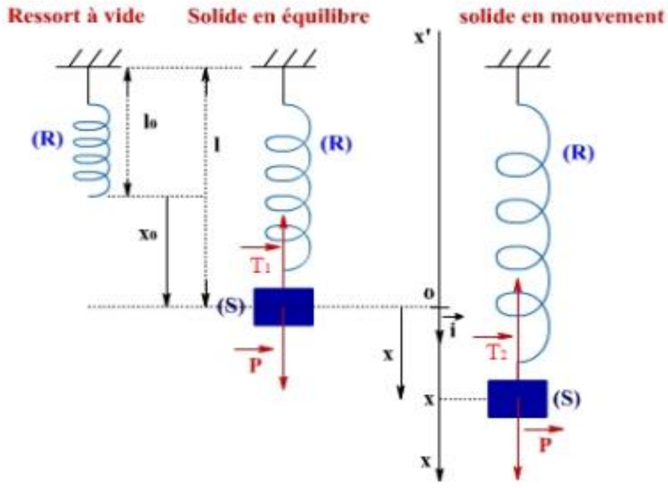
Several methods are used to determine the differential equation representing the motion. Among these methods are: Newton's method, Lagrange's method, the energy method, etc.

In this course, we will use only two methods: **Newton's method** and **Lagrange's method**.

a- Newton's Method

Example 1:

Study of the motion of a harmonic oscillator ; a spring (k) attached to a mass (m). Consider a mass attached to the end of a vertical, mass less spring. This mass moves without friction in the vertical plane. At $t = 0$, the mass is displaced from its equilibrium position by a distance x, then released with zero initial velocity.



Applying Newton's Method

a. At equilibrium $\sum\vec{F} = 0 \rightarrow \vec{P} + \overrightarrow{T1} = 0 \rightarrow \text{P} - \text{T1} = 0$

$$mg - kx_0 = 0 \ \ldots\ldots\ldots\ldots\ldots(1)$$

b. At movement $\sum\vec{F} = m\vec{a} \rightarrow \vec{P} + \overrightarrow{T2} = m\vec{a} \rightarrow \text{P} - \text{T2} = ma$

$$mg - k(x_0 + x) = ma\ .$$

$$mg - kx_0 - kx = m\ddot{x}$$

$$-\,kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$

Finally, we can write the differential equation that represents the motion as:

$$\boldsymbol{\ddot{x} + \frac{K}{m}x = 0}$$

We observe that this is a **second-order linear homogeneous differential equation**.

b. Méthode de Lagrange :

The **Lagrange equation** is used to determine the equation of motion of mechanical systems. It is described by the following equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

Where **L** is the *Lagrangian*, which is an explicit function of the generalized coordinates and generalized velocities:

L = T – U

- **T**: is the total kinetic energy of the system
- **U**: is the total potential energy of the system
- q_i and $\dot{x}_i$: are the generalized coordinates and generalized velocities
- F_i: are the generalized forces associated with q_i

In the case of a **conservative system with one degree of freedom**, the equation of motion reduces to:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Example

Consider the same system (mass–spring). Using the **Lagrange method**, write the equation of motion and determine the natural angular frequency (ω_0).

- **Lagrangian:** $\mathbf{L = T - U}$
- **Kinetic energy:** $\mathbf{T = (1/2)\cdot m\cdot v^2 = (1/2)\cdot m\cdot \dot{x}^2}$
- **Potential energy:** $\mathbf{U = (1/2)\cdot k\cdot x^2}$

So the Lagrangian becomes:

$\mathbf{L = (1/2)\cdot m\cdot \dot{x}^2 - (1/2)\cdot k\cdot x^2}$

Lagrange's equation:

$\mathbf{d/dt\,(\partial L/\partial \dot{x}) - \partial L/\partial x = 0}$

Now compute the derivatives:

- $\partial L/\partial \dot{x} = \mathbf{m\cdot \dot{x}} \quad \Rightarrow \quad d/dt(\partial L/\partial \dot{x}) = \mathbf{m\cdot \ddot{x}}$
- $\partial L/\partial x = \mathbf{-k\cdot x}$

Substitute into the Lagrange equation:

$\mathbf{m \cdot \ddot{x} + k \cdot x = 0}$

After performing the derivations, we obtain the following equation: $m\ddot{x} + kx = 0$
If we divide this equation by **m**, we get the equation of motion:

$$\ddot{x} + \frac{K}{m}x = 0 \text{ ....(2)}$$

Solution of the Equation of Motion

The differential equation of the harmonic oscillator admits the following sinusoidal solution:

$$x = A\cos(\omega_0 t + \varphi)$$

The amplitude **A** and the phase angle **φ** depend on the initial conditions. To determine their values, two initial conditions are needed (usually $x(t_0)$ and $\dot{x}(t_0)$). Therefore, these constants can vary depending on the initial conditions.
The velocity of the mass is given by the first derivative of the position **x(t)**:

$\dot{x} = -Aw_0 \sin(w_0 t+\varphi)$

$\ddot{x} = -Aw_0^2 \cos(w_0 t+\varphi)$

So $\ddot{x} = -w_0^2 x$

We obain the following equation : $\ddot{x} + w_0^2 x = 0$ ....(3)

By comparing equation (2) with this equation, we deduce that:

$$\omega_0^2 = \frac{k}{m} \quad \Rightarrow \quad \omega_0 = \sqrt{k\,m}$$

ω_0 is the natural (free) angular frequency.

Energy of a Harmonic Oscillator

The energy of a harmonic oscillator is the sum of its kinetic and potential energies:

E = T + U

- The **translational kinetic energy** of a body with mass **m** and velocity **v** is:

$T\_transl = (1/2) \cdot m \cdot v^2$

- The **rotational kinetic energy** of a body with moment of inertia I_Δ about an axis Δ and angular velocity $\dot{\theta}$ is:

$T_{rot} = (1/2)\cdot I_\Delta \cdot \theta^2$

- The **potential energy** of a mass **m** in a constant gravitational field **g** is:

$U = m\cdot g\cdot h$

(or **$U = -m\cdot g\cdot h$** in the case of descending a height **h**)

- The **potential energy** of a spring with stiffness **k** during a deformation **x** is:

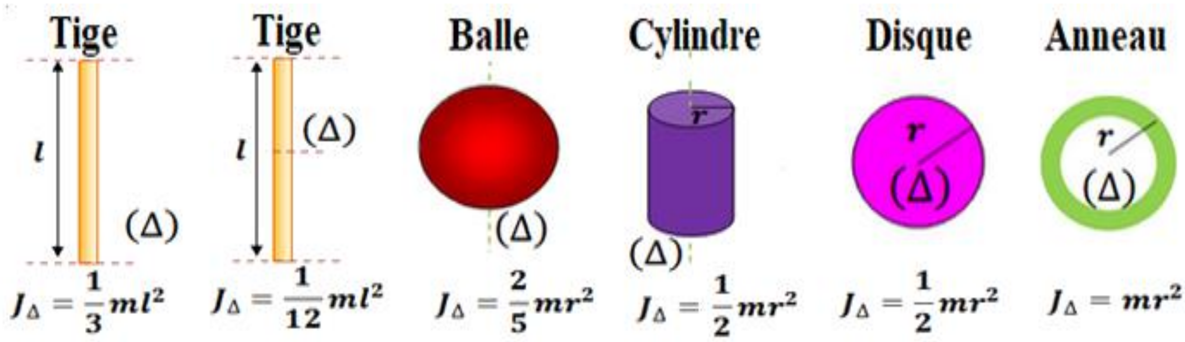
$U_{spring} = (1/2)\cdot k\cdot x^2$

- The **potential energy** of a torsional spring with stiffness **k** during an angular deformation **θ** is:

$U_{torsion} = (1/2)\cdot k\cdot \theta^2$

Systèmes mécaniques		**Système électrique**
Translation	**Rotation**	**Circuit RLC**
Déplacement : x	Angle : θ	Charge : q
Vitesse : $\dot{x}$	Vitesse angulaire : $\dot{\theta}$	Courant : $\dot{q}$
Accélération : $\ddot{x}$	Accélération angulaire : $\ddot{\theta}$	Variation de courant : $\ddot{q}$
Constante de raideur : k	Constante de torsion : C	Inverse de la capacité : 1/c
Masse : m	Moment d'inertie : I	Inductance : L
Coefficient d'amortissement : δ	Coefficient d'amortissement : δ	Résistance : R
Energie cinétique : $\frac{1}{2}m\dot{x}^2$	$\frac{1}{2}I\dot{\theta}^2$	Energie magnétique : $\frac{1}{2}L\dot{q}^2$
Energie potentielle : $\frac{1}{2}kx^2$	$-mgh$	Energie électrique : $\frac{1}{2c}q^2$

Note: The inertia of a body depends on its dimensions, its mass, and its axis of rotation. The figure shows the moment of inertia of different bodies.



Huygens' Theorem

The moment of inertia varies depending on the axis of rotation. If $\mathbf{I_0}$ is the moment of inertia of a body with mass **m** when the axis of rotation passes through the center of mass, and $\mathbf{I_\Delta}$ is the moment of inertia when the axis of rotation is Δ, at a distance **d** from the center of mass, then Huygens' theorem gives the moment of inertia by the following formula:

$$\mathbf{I_\Delta = I_0 + md^2}$$