Linear programming (LP)

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Dr. Guettiche Mourad

I Linear programming

Mathematical programming is the transcription of a practical problem from its verbal expression into mathematical terms and seeking its optimal solution.

The optimal solution is the one that minimizes or maximizes a function that has the economic meaning of a cost or profit and that depends linearly on the variable to be determined.

I Linear programming

If the constraints imposed on the problem depend linearly on its variables, the problem is referred to a linear programming problem.

A linear program is therefore a problem that consists of minimizing or maximizing a linear function (objective function, goal, economic function, etc.) under linear constraints (linear equations or inequalities).

II. History of linear programming

Linear programming was first introduced by Leonid Kantorovich in 1939.

He developed the first linear programming problems that were used by the military during World War II to reduce expenses.

The method was kept secret because of its use in wartime strategies.

II. History of linear programming

In 1947, George B. Dantzig published the simplex method, while John von Neumann developed the theory of duality.

This algorithm (the simplex algorithm) was implemented on early computers.

II. History of linear programming

After World War II, many industries began to adopt linear programming to optimize their manufacturing plans.

In 1984, Indian mathematician KARMARKAR proposed a new method developed at Bell Labs that enabled very large linear problems to be solved using an approach internal to the polyhedron.

II. Mathematical expression of PLs

Objective or economic function generally subject to a number of constraints in the form of linear equations.

$$Z = c_1x_1+c_2x_2+\dots+c_nx_n.$$

$c_1,c_2,\dots,c_n$: are called the coefficients of the objective function.

II. Mathematical expression of PLs

Constraints can be grouped as either equalities or inequalities, categorised according their meaning.

$$\begin{cases} a_{i1}\,x_1 + a_{i2}\,x_2 + \cdots + a_{in}\,x_n \leq b_i & i = 1 \ldots k \\ a_{i1}\,x_1 + a_{i2}\,x_2 + \cdots + a_{in}\,x_n = b_i & i = k+1 \ldots r \\ a_{i1}\,x_1 + a_{i2}\,x_2 + \cdots + a_{in}\,x_n \geq b_i & i = r+1 \ldots m \end{cases}$$

II. Mathematical expression of PLs

The coefficients a_{ij} are called the constraint coefficients.

The variables x_j are called decision variables.

The coefficients a_{ij} , b_i, c_j constitute the problem data (Input).

The solution of the LP consists of determining the values of x_j(outputs) that optimize the objective function.

II. Mathematical expression of PLs

$x_1 \geq 0,\ x_2 \geq 0, \ldots, x_n \geq 0.$

These constraints are called the non-negativity constraints of the decision variables.

III. Mathematical formulation of PLs

Step 1

- Identify the decision variables of the given problem.

Step 2

- Identify the constraints of the problem and express them using a system of linear equations and/or inequalities.

Step 3.

- Identify the objective function and present it in linear form with respect to the decision variables.

III. Mathematical formulation of PLs

A factory manufactures two products, A and B, from a raw material C.
Three units of C are needed for product A. Four units of C are needed for product B.
There are 15 units of C available.
The selling price per unit for product A is 15 dinars and that for product B is 10 dinars.

III. Mathematical formulation of PLs

The problem for the company's management is to find the maximum number of products A and B to manufacture in order to maximize profit, while respecting the constraints on the quantity of C.

III. Mathematical formulation of PLs

Step 1: Variable definition

x_1 : Number of A product units to be manufactured,

x_2 : Number of B product units to be manufactured.

$x_1, x_2 \geq 0.$

III. Mathematical formulation of PLs

Step 2: Writing constraints

$3x_1$: is the number of units of C required to manufacture x_1 unit of A,

$4x_2$: is the number of units of C required to manufacture x_2 unit of B.

Therefore, we have $3x_1+4x_2 \leq 15$.

III. Mathematical formulation of PLs

Step 3: The objective function:

Max $z=15x_1+10x_2$

Subject to:

$3x_1+4x_2 \leq 15.$

$x_1,\ x_2 \geq 0.$

IV Linear program solving

Solving a linear program (LP) means determining the numerical values that the decision variables must take to maximize or minimize the objective function, while respecting the constraints of the problem.

Two methods will be used:

- The graphical method
- The simplex method

V. The graphical method

- The graphical method is used to solve linear programming problems with two decision variables.
- The method has limitations for three variables, it becomes complicated to present.
- The method is impractical for more than three variables.

V.1 Graphical solution of a linear problem

Solving a linear problem involves answering the following questions:

Q_1
- How can we define the set of feasible solutions?

Q_2
- Is there an optimal solution?

Q_3
- How can we determine if it exists?

V.1.1 Graphical representation of constraints

In this graphical approach, we begin by drawing the lines that represent equality constraints and the lines that delimit inequality constraints.

For example, the inequality corresponding to the non-negativity constraint, $x_2 \geq 0$, divides the plane into two half-planes relative to the horizontal axis x_1: the upper plane and the lower plane.

V.1.2 Visualization of feasible solutions

The geometric shape that defines the set of feasible solutions is a convex polyhedron.

The extreme points of this polyhedron are obtained by the intersection of the lines representing the limits of the constraints of the LP.

Property: The set of feasible solutions for an LP is a convex polyhedron, whose edges are obtained by the intersection of lines, planes, or hyperplanes representing the constraints of the LP.

V.1. 3 Determination of the optimal solution

The optimal solution for a LP is determined by identifying the vertices of the convex polyhedron that delimits the set of feasible solutions.

Then, by evaluating and comparing the values of the objective function at these peaks, we determine the optimal solution.

Example :

Min/max $Z = 3x_1 + 4x_2$

$2x_1+3x_2\leq 6$

$x_1+2x_2\geq 2$

$x_1\geq 0$; $x_2\geq 0$

VI. Special case

1. Degenerate solution.
2. Unbounded solution.
3. Multiple solution.
4. No feasible solution.

VI.1 Degeneration

The problem of degeneration occurs when at least three constraints converge at the same point.

In principle, an extreme point is the result of the intersection of two constraints.

The case of degeneration causes a certain degree of indeterminacy, as an extreme point becomes a point of intersection for more than two constraints.

Example:

Max $Z = X_1 + X_2$

Sous les contraintes

$8X_1 + 4X_2 \leq 32$

$4X_1 + 6X_2 \leq 24$

$X_1 \leq 3$

$X_1 \geq 0 \; ; \; X_2 \geq 0$

VI.2 Multiple solutions

The optimal solution may not be unique, as illustrated in the following example:

Max $Z = 2X_1 + 4X_2$

subject to constraints

$4X_1 + 8X_2 \leq 12$

$6X_1 + 5X_2 \leq 30$

$X_1 \geq 0, X_2 \geq 0$

IV.3. Unbounded solution

A feasible unbounded region is one in which at least one of the decision variables can take infinite values.

Max $Z = X_1 + X_2$

subject to constraints

$X_1 - X_2 \leq 2$

$-X_1 + X_2 \leq 2$

$X_1 \geq 0, X_2 \geq 0$

IV.3. No feasible solution

Max $Z = 10X_1 + 3X_2$

subject to constraints

$X_1 + X_2 \leq 3$

$2X_1 + 2X_2 \geq 9$

$X_1 \geq 0,\ X_2 \geq 0$

VII. Discussion

Geometric reasoning is always feasible for $n = 2$ and, in a pinch, for $n = 3$. However, industrial problems can involve hundreds of variables, or even more.

The optimum, if it exists, of a linear programme is achieved at a peak of the polyhedron. This could lead to enumerating the vertices of the polyhedron.

However, as this number increases rapidly with m and n, enumeration is impractical for industrial-scale problems.

VIII. Complexity classes

non-deterministic polynomial time (NP): The optimality of a proposed solution can be verified in polynomial time.

If there is an algorithmic solution for which the complexity is bounded by a polynomial, the NP problem is said to be polynomial.

Otherwise, if we have succeeded in reducing it to a well-known NP-complete problem, the problem in this case is NP-complete.

VIII. Complexity classes

If the polynomial verification of the solution property is not verified, the problem is said to be NP-hard.