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Algebra I: Course Notes.
Chapter 2. Sets and Binary Relations

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Life is good only for two things, discovering mathematics and teaching mathematics.

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This chapter is divided into two main parts.

Part I – Sets

This part explains what a set is—a collection of objects. It introduces basic ideas such as membership, inclusion, equality, and the difference between finite and infinite sets. It also presents the main operations on sets: complement, intersection, union, difference, and symmetric difference, with clear definitions and diagrams. Other topics include the Cartesian product of sets, the union and intersection of families of sets, and partitions (dividing a set into disjoint parts). The part ends with the main properties of set operations, like commutativity, associativity, distributivity, and De Morgan's laws.

Part II – Binary Relations

This part studies how elements of sets can be related to each other. It defines binary relations, and explains their graph, domain, range, inverse, and composition. It then describes the main properties of relations: reflexive, symmetric, antisymmetric, and transitive. Special sections cover equivalence relations and order relations. The first explains how equivalence relations form equivalence classes and quotient sets, while the second introduces order relations, including partial and total orders, and key ideas such as upper and lower bounds, maximum, minimum, supremum, and infimum.

Summary: In short, this chapter gives a clear and structured introduction to sets and binary relations, two basic tools for studying algebra and logic.

Part I

Sets

2.1 Definitions and notations

Definition 1

1. A set is a collection of objects; these objects are called the elements or members of the set, and they are considered without order and without repetition (this means that the order in which elements are written and the repetition of an element does not change the nature of a set), i.e. the following lists describe the same set

$$\{1, 1, 2, 3\} = \{1, 2, 3\} = \{3, 2, 1\}.$$

2.A set is usually denoted by capital letters A, B, C,... etc. while its members (or elements or objects) are denoted by by small (or lowercase) letters such as a, b, c,....etc.

Example 2 ***Standard Notations for Sets of Numbers.***

• *The set of natural numbers (positive integers), $\mathbb{N} = \{0, 1, 2, ...\}$.*

• *The set of integer numbers, $\mathbb{Z} = \{..., -2 - 1, 0, 1, 2, ...\}$*

• *The set of rational numbers $\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\right\}$ (p is the numerator and q is the denominator of the fraction $\frac{m}{n}$).*

• *The set of real numbers $\mathbb{R}$.*

• *The set of complex numbers $\mathbb{C}$.*

• *The set of positive integers $\mathbb{Z}^+$.*

• *The set of negative integers $\mathbb{Z}^-$.*

• *The set of positive rational numbers $\mathbb{Q}^+$.*

• *The set of negative rational numbers $\mathbb{Q}^-$.*

• *The set of positive real numbers $\mathbb{R}^+$.*

• *The set of negative real numbers $\mathbb{R}^-$.*

• *S^* (pronounced "S star") generally denotes the set of non-zero elements of a set S. (The star indicates the absence of 0.). For example:*

• *The set of non-zero natural numbers $\mathbb{N}^* = \{1, 2, ...\}$.*

• *The set of non-zero integer numbers $\mathbb{Z}^*$.*

• *The set of non-zero rational numbers $\mathbb{Q}^*$.*

• *The set of non-zero real numbers $\mathbb{R}^*$.*

• *The set of non-zero complex numbers $\mathbb{C}^*$.*

The letter $\mathbb{Z}$ comes from "zahl" (German for "number") and "$\mathbb{Q}$" comes from "quotient".

Remark 3 *In set theory, a set can be defined in two fundamental ways: by extension and by comprehension:*

1. ***Definition by Extension (also called Roster Form):*** *In this method, all the elements of the set are listed one after another and separated by commas (explicit form). The list is then enclosed in curly brackets or braces { and }, which indicate the beginning and end of the set description. This method is useful when the set has a small number of elements. For example, the set of prime numbers less than* 10 *can be defined by extension as*

$$S = \{2, 3, 5, 7\}.$$

2. ***Definition by Comprehension (also called Set Builder Notation):*** *In this method, the elements of a set are represented by a variable x followed by a property $\mathcal{P}(x)$ that each element must satisfy. This representation is enclosed in curly brackets, as follows:*

$$S = \{x \in E : \mathcal{P}(x)\}.$$

For example, the set of all even integers can be defined by comprehension as:

$$S = \{x \in \mathbb{Z} : x \textit{ is even}\}.$$

The membership relation describes the relationship between an element and a set. It is denoted by the symbol $\in$.

Definition 4 ***(Membership relation)***

Let A be a set, then

1. If an element x belongs to a set A, we write $x \in A$. This is read as "x is an element of A" or "x belongs to A." 'x is an object of A'.

2. If an element x does not belong to a set A, we write $x \notin A$. This is read as "x is not an element of A" or "x does not belong to A, or "x is not an object of A".

3. The greek symbol $\in$ is called the membership symbol

Example 5 *If $A = \{1, 2, 3\}$ and $x_1 = 2, x_2 = 4$, then $2 \in A$, meaning 2 is an element of the set A and $4 \notin A$ meaning 4 is not an element of the set A.*

Definition 6 ***(Types of sets)***

1. ***Empty Set****: A set with no element is called the empty (or null, or void) set, and is denoted by the symbol*

ϕ (read as phi) or by two empty braces $\{\}$. Therefore, we have

$$\forall x, x \notin \phi.$$

This means that for any element x, x does not belong to the empty set.

*2. **Singleton set**: A Set containing only one element is called a singleton set (It is a set with only one member). Therefore a singleton is written $\{x\}$.*

*3. **Pair Set:** A pair set is a set that has exactly two elements. It is written as $\{x, y\}$, where x and y are the two distinct elements of the set.*

Example 7

1. If $A = \{x \in \mathbb{R} : x^2 = -1\}$ has no element, i.e., $A = \phi$ is the empty set.

2. If $A = \{x \in \mathbb{R} : x \neq x\} = \phi$.

3. $A = \{x \in \mathbb{N} : x^3 = 27\}$ is a singleton set with a single element $\{3\}$.

Definition 8 ***(Finite, Infinite Sets and Cardinality)***

1. A set is said to be finite if it has a finite number of elements. The number of elements in a finite set A is called its cardinality, denoted by $|A|$ or $Card(A)$.

2. A set which contains an infinite number of elements is called an infinite set. In other words, A set which is not finite, is called an infinite set (we cannot count the number of elements in the set).

3. The cardinality of an infinite set A is $Card(A) = \infty$.

Example 9

1. An empty set ϕ is a finite set with cardinality equal to zero.

2. $A = \{p \in \mathbb{N} : p \text{ is a prime number}\}$ is an infinite set.

3. The set $\mathbb{N}$ of natural numbers is infinite. Similarly, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, the set of rational numbers, of integers, of real numbers and complex numbers, respectively, are also infinite sets.

2.2 Relationships between sets

There are two basic relationships between sets: the inclusion and the equality.

2.2.1 Inclusion Relationship

Definition 10 *(The inclusion relationship)*

1. Let A and B be two sets. We say that A is included in B (or that A is a subset of B, or that A is a part of B, or that B contains A) if and only if every element of A is also an element of B. This relationship is denoted as $A \subset B$ and can be expressed mathematically as

$$(A \subset B) \Longleftrightarrow \forall x, (x \in A \Longrightarrow x \in B).$$

2. The symbol $\subset$ is the set inclusion symbol.

3. If A is a subset of B and $A \neq B$, then we say that A is a proper subset of B.

4. If A is not included in B (or A is not a subset of B), we denote this as $A \not\subset B$. This means that there exists at least one element of A that does not belong to B. Thus, we can express this as

$$A \not\subset B \Longleftrightarrow \exists x : (x \in A \text{ and } x \notin B).$$

Example 11 *1. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5\}$ and $C = \{2, 4, 6\}$. Then we have $B \subset A$ because*

$$\forall x, (x \in B \Longrightarrow x \in A).$$

This means that for each element of A, this element also belongs to B but $C \not\subset A$ because

$$\exists x = 6 : (6 \in C \text{ et } 6 \notin A).$$

2. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

3. $\mathbb{Q} \not\subset \mathbb{Z}$ because $\frac{1}{3} \in \mathbb{Q}$ et $\frac{1}{3} \notin \mathbb{Z}$.

Proposition 12 *Basic facts about inclusion*

1. For every set S, $S \subset S$. In other words, every set is a subset of itself.

2. For every set S, $\phi \subset S$. In other words, the empty set is a subset of every set.

3. For the given sets A, B and C, we have if $A \subset B$ and $B \subset C$, then $A \subset C$.

Proof.

1. Since every element of S is an element of S, therefore $S \subset S$.

2. Suppose on the contrary $\phi \not\subset S$. Then there exists $x \in \phi$ such that $x \notin S$. This is absurd as ϕ

does not contain any element. Our assumption is wrong, so that, $\phi \subset S$.

3. Assume $A \subset B$ and $B \subset C$. Let $x \in A$, since $A \subset B$, by definition of subset, $x \in B$. Now, since $B \subset C$ and $x \in B$, it follows that $x \in C$ (by the definition of subset again). Therefore, for every $x \in A$, we have $x \in C$, we conclude that $A \subset C$. ■

Remark 13 ***(Difference Between Membership and Inclusion in Set Theory)***

1. Membership: Membership is denoted by the symbol $\in$ and describes the relationship between a single element and a set. When an element x belongs to a set S, we write $x \in S$. This indicates that x is one of the elements within S.

2. Inclusion: Inclusion is represented by the symbol $\subset$ and refers to the relationship between two sets. Saying that a set A is included in a set B ($A \subset B$) means that all elements of A are also elements of B. Writing $A \subset B$ implies that A can be exactly equal to B or a proper subset of B.

3. In summary, the membership relation tells us if an element is part of a set, while the inclusion relation tells us if one set is a subset of another. In other words, the symbol $\in$ connects an element with a set (membership), whereas the symbol $\subset$ connect two sets (inclusion).

Definition 14 ***(Power Set)***

1. The set of all subsets of a given set S is called the power set of S (or set of parts of S). It is denoted by $\mathcal{P}(S)$. Thus, every element of power set S is a set. Mathematically,

$$\mathcal{P}(S) = \{X : X \subset S\}.$$

In other words,

$$X \in \mathcal{P}(S) \Longleftrightarrow X \subset S.$$

.2. Note that $\phi \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$ for all sets S.

Example 15

1. If $S = \{a\}$, then the only subsets of S are the empty set and S itself. Therefore,

$$\mathcal{P}(S) = \left\{\phi, S\right\}.$$

2.If S has two elements, say $S = \{a, b\}$ with $a \neq b$, then the power set is

$$\mathcal{P}(S) = \left\{\phi, \{a\}, \{b\}, S\right\}.$$

Proposition 16 *If S is a finite set with n elements ($Card(S) = n$), then the power set $\mathcal{P}(S)$ is also finite, and its cardinality is given by*

$$Card(\mathcal{P}(S)) = 2^n.$$

2.2.2 Equality Relationship

Another fundamental relationship between sets is, in fact, the concept of set equality.

Definition 17 ***(Equality of Two Sets)***

Let A and B be two sets.

1. We say that A and B are equal (or identical) denoted as $A = B$ (read as "A is equal to B" or "A is identical to B") if they have exactly the same elements. This means that every element of A is also an element of B, and every element of B is also an element of A. Logically, this means

$$A = B \Longleftrightarrow \forall x, (x \in A \Longleftrightarrow x \in B).$$

In other words,

$$A = B \Longleftrightarrow (A \subset B \text{ and } B \subset A) \quad \text{(Principle of Double Inclusion)}.$$

2. If A and B do not contain the same elements, we say they are distinct, denoted as $A \neq B$ (read as "A is different from B" or "A is distinct from B." This can be expressed as

$$A \neq B \Longleftrightarrow \exists x, (x \in A \text{ and } x \notin B) \text{ or } \exists x, (x \notin A \text{ and } x \in B).$$

Example 18

1. If $A = \{x \in \mathbb{R} : x^2 - 3x + 2 = 0\}$ and $B = \{1, 2\}$, then we have $A = B$. To show that $A = B$, we must demonstrate that both sets contain exactly the same elements. We have A is the set of solutions to the equation $x^2 - 3x + 2 = 0$, then the solutions to $x^2 - 3x + 2 = 0$ are $1, 2$. Therefore, we have $A = B$.

2.3 Operations on Sets (Set Operations)

Let us now study sets obtained by applying operations on sets. We will cover five operations here, namely,complement of sets, intersection of sets, union of sets, Difference and symmetric difference of sets. We describe below the basic operations on sets.

2.3.1 Complement of Sets

Definition 19 ***(Complement of a set)***

The complement of a set A with respect to a universal set E is the set of all elements in E that are not in E. It is denoted by C_E^A or $\overline{A}$ or sometimes ${}^{C}A$. Symbolically, the complement of a set A is expressed as

$$C_E^A = \{x \in E : x \notin A\}.$$

In other words,

$$x \in C_E^A \Longleftrightarrow x \notin A.$$

This means that an element x belongs to the complement of set A with respect to the universal set E if and only if x is not an element of A.

• We can represent the complement of A with respect to the set E using the following diagram (Shaded region represents C_E^A).

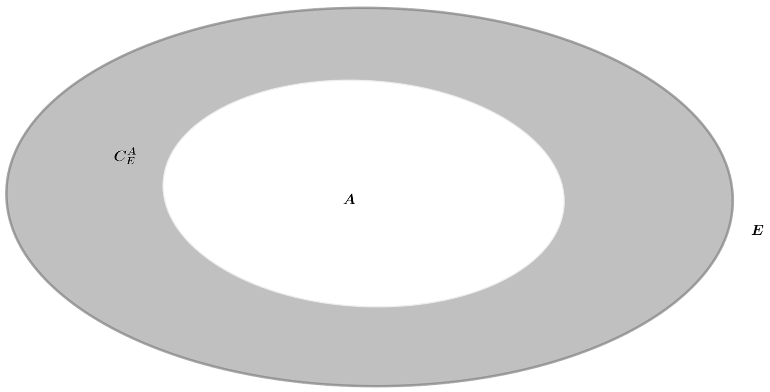

Figure 2.1: Complement of a set

Example 20

1. Given the universal set $E = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ *and the subset* $A = \{2, 4, 6, 8\}$. *Then the complement of the set A is*

$$C_E^A = \{x \in E : x \notin A\} = \{0, 1, 3, 5, 7\}.$$

2. The complement of the subset $\mathbb{P}$ of even integers within the set $\mathbb{N}$ of natural numbers is the set of odd integers $\mathbb{I}$, denoted by

$$C_{\mathbb{N}}^{\mathbb{P}} = \mathbb{I}.$$

2.3.2 Intersection of sets

Definition 21 ***(Intersection of sets)***

Let E be a set and A and B be two subsets of E.

1. The intersection of A and B is the set of all elements which are both in A and B (i.e. set of the common elements in A and B). It is denoted by $A \cap B$ and is read as "A intersection B" or "A cap B". Symbolically, it is expressed as

$$A \cap B = \{x \in E : x \in A \wedge x \in B\}.$$

In other words,

$$x \in A \cap B \Longleftrightarrow x \in A \wedge x \in B.$$

This means that an element x belongs to the intersection of sets A and B if and only if x is an element of both A and B.

2. If $A \cap B = \phi$, we say that the sets A and B are disjoint (if their intersection is the empty set i.e, sets have nocommon elements).

• We can represent the intersection of the two sets A and B with the following diagram (Shaded region represents $A \cap B$).

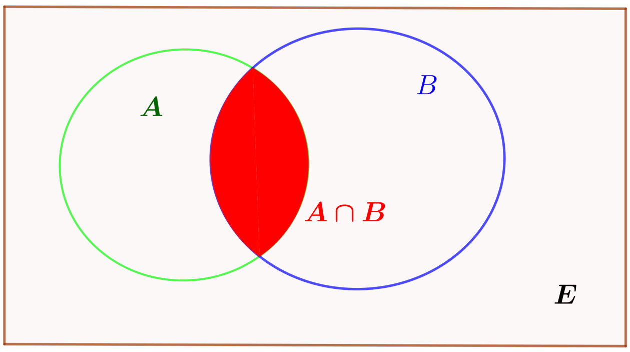

Figure 2.2: Intersection of sets

Example 22 *For example $A = \{1,2,3\}$ and $B = \{2,1,5,6\}$, then $A \cap B = \{1,2\}$. Again if $A = \{1\}$ and $B = \{5\}$ then $A \cap B = \{\}$ or ϕ.*

2.3.3 Union of Sets

Definition 23 ***(Union of Sets)***

Let E be a set and A and B be two subsets of E.

*1. **The union** of A and B is the set of all elements that are either in A or in B (or in both). This set is denoted by $A \cup B$, and is read as 'A union B' or " the union of A and B". Symbolically, it is expressed as*

$$A \cup B = \{x \in E : x \in A \vee x \in B\}.$$

In other words,

$$x \in A \cup B \Longleftrightarrow x \in A \vee x \in B.$$

This means that an element x belongs to the union of sets A and B if and only if x is an element of A or x is an element of B.
2. The union $A \cup B$ is called a disjoint union when $A \cap B == \phi$ (i.e, when the intersection of the two sets is empty, meaning that A and B share no common elements.)

- We can represent the union of the two sets A and B with the following diagram (Shaded region represents $A \cup B$).

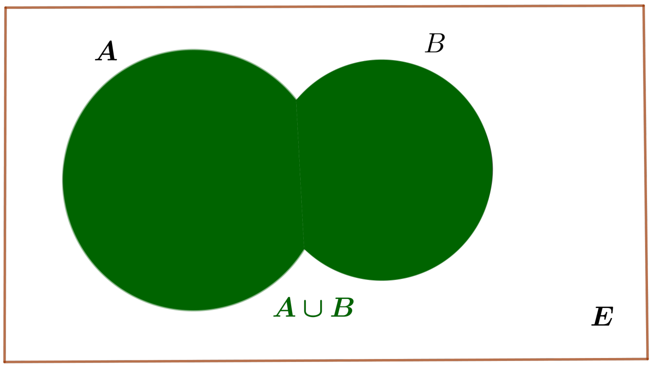

Figure 2.3: Union of Sets

Example 24

1.

$$\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$$

2.

$$\{x \in \mathbb{R} : x > 0\} \cup \{x \in \mathbb{R} : x > 1\} = \{x \in \mathbb{R} : x > 0\}.$$

2.3.4 Difference of sets

Definition 25 ***(Difference of sets)***

Let E be a set and A and B be two subsets of E.

1. The difference of A and B, denoted as $A - B$ or $A \backslash B$, is the set of elements that belong to A but not to B is read as "A minus B" or "the difference of A and B." Symbolically, it is expressed as

$$A \backslash B = \{x \in E : x \in A \wedge x \notin B\} = A \cap C_E^B \subset A.$$

In other words,

$$x \in A \backslash B \Longleftrightarrow x \in A \wedge x \notin B.$$

This means that an element x belongs to the difference $A \backslash B$ if and only if x is an element of A and x is not an element of B.

Example 26 *For example, if $A = \{4, 5, 6, 7, 8, 9\}$ and $B = \{3, 5, 2, 7\}$, then $A \backslash B = \{4, 6, 8, 9\}$ and $B \backslash A = \{3, 2\}$. From this example it is clear that $A \backslash B \neq B \backslash A$. So, the difference of sets is not a commutative operation.*

• We can represent the difference of the two sets A and B using the following diagram: (Shaded region represents $A \backslash B$ and $B \backslash A$)

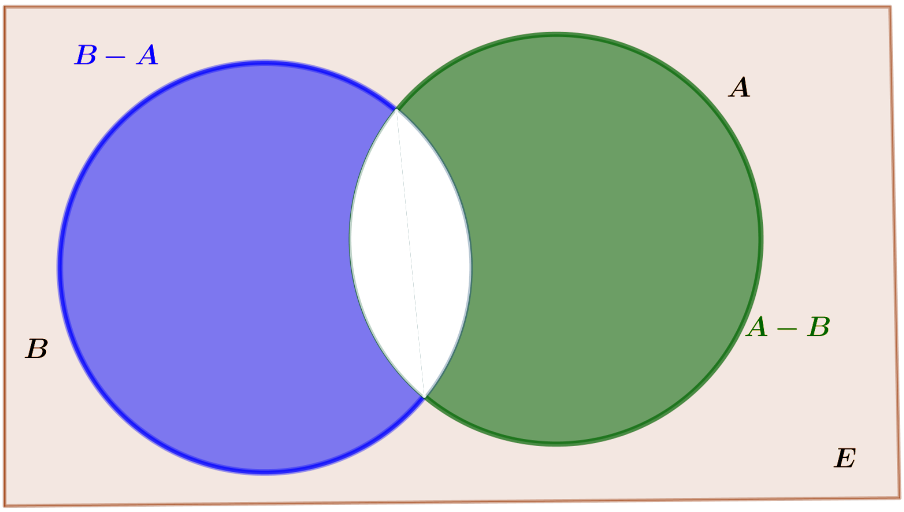

Figure 2.4: Difference of sets

2.3.5 Symmetric difference of sets

Definition 27 ***(Symmetric difference of sets)***

Let E be a set and A and B be two subsets of E.

1. The symmetric difference of sets A and B is defined as the set of all elements that belong to either A or B, but not to both (i.e., those that are not in their intersection). It is denoted by $A \triangle B$, which is read as "A delta B." Symbolically, this can be expressed as

$$A \triangle B = (A \backslash B) \cup (B \backslash A) = \{x \in E : x \in A \backslash B \vee x \in B \backslash A\}.$$

Example 28 *For example $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 5, 6, 7\}$, then $A \backslash B = \{1, 2, 4\}$,and $B \backslash A = \{6, 7\}$, so*

$$A \triangle B = (A \backslash B) \cup (B \backslash A) = \{1, 2, 4, 6, 7\}.$$

- We can represent the symmetric difference of the two sets A and B using the following diagram: (Shaded region represents $A \triangle B$).

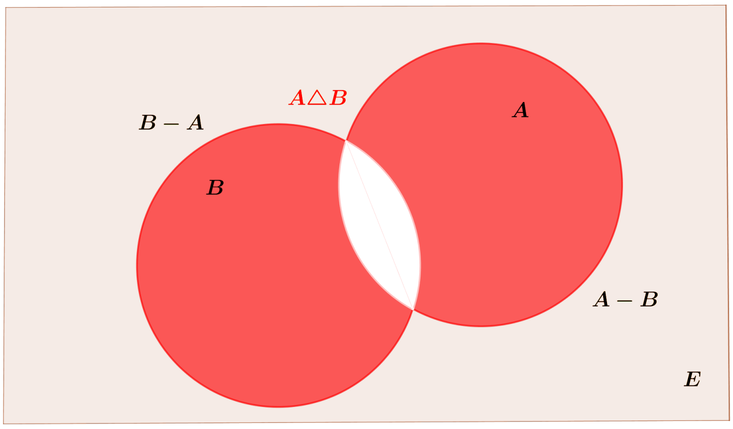

Figure 2.5: Symmetric difference of sets

2.4 Cartesian Product of Two Sets

To introduce the notion of the Cartesian product, we first need to define the concept of an ordered pair.

Definition 29 ***(Ordered Pair)*** *Let A and B be two non-empty sets, then.*

1. An ordered pair (with first element from A and second element from B) is a single pair of objects, denoted by (a, b), with $a \in A$ and $b \in B$ and an implied order. This means that for two ordered pairs to be equal, they *must contain exactly the same objects in the same order. That is,*

$$(a,b)=(c,d)\Longleftrightarrow a=c \text{ and } b=d.$$

2. The objects in the ordered pair are called the coordinates of the ordered pair. In the ordered pair (a,b), a is the first coordinate and b is the second coordinate.

Definition 30 ***(Cartesian product)***

1. Let A and B be two non-empty sets, then

(a). The Cartesian product of A and B is the set of all possible ordered pairs (a,b), where the first element a belongs to A and the second element b belongs to B. This product is denoted by $A\times B$ (read as "A cross B") and is formally defined as

$$A\times B=\{(a,b): a\in A \text{ and } b\in B\}.$$

(b). If $A=B$, we denote this Cartesian product as A^2 (i.e, $A^2=A\times A$).

(c). If $A=\phi$ or $B=\phi$, then $A\times B=\phi$. In other words, the Cartesian product of any set with the empty set results in the empty set.

We can extend the definition of $A\times B$ to define the Cartesian product of n sets $A_1,A_2,...,A_n$ as follows.

Definition 31 *Let $A_1,A_2,...,A_n$ be non-empty sets. Then*

(a) The Cartesian product of the sets $A_1,A_2,...,A_n$ is denoted by $A_1\times A_2\times ...\times A_n=\prod_{i=1}^{n}A_i$ and is defined as the set of all ordred $n-$tuples $(a_1,a_2,...,a_n)$ (is called an $n-$tuple.where) $a_i\in A_i$ for all $i\in\{1,2,...,n\}$. In other words,

$$A_1\times A_2\times ...\times A_n=\{(a_1,a_2,...,a_n): a_1\in A_1, a_2\in A_2,...,a_n\in A_n\}.$$

(b) In particular, for every $n\in\mathbb{N}^*$, we define

$$E^n=\underbrace{E\times E\times ...\times E}_{n \text{ times}}.$$

Example 32

1. If $A=\{1,2\}$ and $B=\{a,b,c\}$ then

$$A\times B=\{(1,a)(1,b),(1,c),(2,a),(2,b),(2,c)\},$$

and

$$A \times A = A^2 = \{(1,1)(1,2),(2,1),(2,2)\}.$$

2.

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}.$$

Proposition 33 *(Number of elements in the Cartesian product of finite sets)*

1. Let A and B be two finite sets. Their Cartesian product $A \times B$ is also finite, and we have

$$Card(A \times B) = Card(A) \cdot Card(B).$$

2. If $A_1, \ldots, A_n$ are finite sets, then their Cartesian product is also finite, and we have

$$Card(A_1 \times A_2 \times ... \times A_n) = Card(A_1) \cdot Card(A_2) \cdot ... \cdot Card(A_n).$$

2.5 The Union and Intersection of a Family of Sets

It is possible to generalize the union to a finite number of sets: one can reduce the case to that of two sets through successive binary unions, and the associativity of union guarantees that the order does not matter. The same applies to intersection. Additionally, these operations can also be generalized to a family of sets that is not necessarily finite.

Definition 34 *Suppose I is a set, called the index set, and with each $i \in I$ we associate a set A_i. We call $(A_i)_{i \in I}$ an indexed family of sets.*

1. The intersection of the family $(A_i)_{i \in I}$ is denoted by $\bigcap\limits_{i \in I} A_i$. It is defined as

$$\bigcap_{i \in I} A_i = \{x \in E, \forall i \in I : x \in A_i\}.$$

Logically, this means

$$x \in \bigcap_{i \in I} A_i \Longleftrightarrow \forall i \in I : x \in A_i$$

That is, the intersection $\bigcap\limits_{i \in I} A_i$ includes all x that are present in every set within the family.

2. The union of the family $(A_i)_{i \in I}$ is denoted by $\bigcup\limits_{i \in I} A_i$. It is defined as

$$\bigcup_{i \in I} A_i = \{x \in E, \exists i \in I : x \in A_i\}.$$

Logically, this means

$$x \in \bigcup_{i \in I} A_i \Longleftrightarrow \exists i \in I : x \in A_i.$$

That is, the union $\bigcup\limits_{i \in I} A_i$ *includes all* x *for which there exists at least one indexed set* A_i *such that* x *belongs to that set.*

2.6 Partitions of Sets

A partition of a set is a division of the set into non-empty, disjoint subsets whose union reconstitutes the original set.

Definition 35 ***(Partition)***

Let E be a non-empty set. A partition of E is a family $\mathcal{F} = (A_i)_{i \in I}$ *of subsets of E that satisfies the following three conditions:*

1. For each $i \in I$, $A_i \neq \phi$ *(each subset is non-empty).*

2. For any $i, j \in I$ *with* $i \neq j$, $A_i \cap A_j = \phi$ *(the subsets are pairwise disjoint or mutually disjoint).*

3. $\bigcup\limits_{i \in I} A_i = E$ *(the union of all* A_i *equals E).*

Example 36

1. The family of sets

$$\mathcal{F} = \{A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\}\}.$$

forms a partition of the set $E = \{1, 2, 3\}$.

2. The family of sets

$$\mathcal{F} = \{A_1 = \mathbb{R}_+^*, A_2 = \mathbb{R}_-^*, A_3 = \{0\}\}$$

forms a partition of the set $\mathbb{R}$.

3. The family of sets

$$\mathcal{F} = \{A_1 = \mathbb{R}_+^*, A_2 = \mathbb{R}_-^*\}$$

does not form a partition of $\mathbb{R}$ *because the union* $\mathbb{R}_+^* \cup \mathbb{R}_-^*$ *does not cover all of* $\mathbb{R}$, *i.e,* $\mathbb{R}_+^* \cup \mathbb{R}_-^* \neq \mathbb{R}$*) (since* $0 \notin \mathbb{R}_+^* \cup \mathbb{R}_-^*$*).*

2.7 Properties of set operations

With the following proposition, we summarize the essential results related to these set operations. The reader is encouraged to establish the following properties as an exercise.

Proposition 37

Let A, B, and C be subsets of a set E. Then:

1. $A \cup B = B \cup A$ (Commutativity of Union).

2. $A \cap B = B \cap A$ (Commutativity of Intersection).

3. $(A \cup B) \cup C = A \cup (B \cup C)$ (Associativity of Union).

4. $(A \cap B) \cap C = A \cap (B \cap C)$ (Associativity of Intersection)

5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributivity of Union over Intersection).

6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributivity of Intersection over Union).

7. $A \cup \phi = A$ et $\phi \cup A = A$ (The empty set is the neutral element for Union).

8. $A \cap \phi = \phi$ et $\phi \cap A = \phi$ (The empty set is the absorbing element for Intersection).

9. $A \cap E = A$ et $E \cap A = A$ (The set E is the neutral element for Intersection).

10. $A \cup E = E$ et $E \cup A = E$ (The set E is the absorbing element for Union).

11. $C_E^{\phi} = E$ et $C_E^E = \phi$ (The set E and the empty set ϕ are symmetric with respect to complementation).

12. $A \cap C_E^A = \phi$, $A \cup C_E^A = E$ et $C_E^{C_E^A} = A$.

13. $C_E^{A \cap B} = C_E^A \cup C_E^B$ et $C_E^{A \cup B} = C_E^A \cap C_E^B$ (De Morgan's Laws).

14. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$ et $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

15. $A \triangle B = B \triangle A$ (Commutativity of Symmetric Difference).

16. $(A \triangle B) \triangle C = A \triangle (B \triangle C)$ (Associativity of Symmetric Difference).

17. $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$ (Distributivity of Intersection with respect to Symmetric Difference)..

19. $A \triangle \phi = A$ et $\phi \triangle A = A$ (The empty set is the neutral element for Symmetric Difference).

20. $A \triangle B = (A \cup B) \backslash (A \cap B)$ (Characterization of Symmetric Difference).

Proposition 38 *Given a family of sets $(A_i)_{i \in I}$ and a set B, all contained within a set E, the following* *identities hold:*

$$1.\ B \cap \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} (B \cap A_i) \qquad \textit{(Distributive Property)}$$

$$2. B \cup \left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} (B \cup A_i) \qquad \textit{(Distributive Property)}$$

$$3. C_E^{\left(\bigcup_{i\in I} A_i\right)} = \bigcap_{i\in I} \left(C_E^{A_i}\right) \qquad \textit{(De Morgan's Law)}$$

$$4. C_E^{\left(\bigcap_{i\in I} A_i\right)} = \bigcup_{i\in I} \left(C_E^{A_i}\right) \qquad \textit{(De Morgan's Law)}$$

Corollary 39 *For sets A and B, the following properties hold:*

1. *If $A \subset B$, then $\mathcal{P}(A) \subset \mathcal{P}(B)$.*
2. *$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.*
3. *$\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$.*

Part II

Binary Relations

In this part we shall not discuss the complete theory of binary relations (although this would probably help with functions, later) but we'll concentrate on two types of relation that are important in the daily practice: equivalence relations and order relations.

2.8 Definitions and notations

Definition 40

1. A binary relation from a set E to a set F is a correspondence (or property), denoted $\Re$, that links elements of E to elements of F.

2. If $x \in E$ is linked to $y \in F$ by the relation $\Re$, we say that x is related to y, or that (x, y) satisfies the relation $\Re$, and we write $x\Re y$. Otherwise, we write $x\Re y$ (i.e, x is not related to y by the relation $\Re$).

3. The subset of $E \times F$ that contains all ordered pairs satisfying the relation is called the graph of $\Re$. It is denoted by Γ and is defined as

$$\Gamma = \{(x, y) \in E \times F : x\Re y\}.$$

4. If $E = F$, we say that $\Re$ is a binary relation defined on the set E.

5. If $\Re$ is a relation from E to F, then

(a). The domain of $\Re$ (or the set of definition of $\Re$) is the set of all elements in E that are related to at least one element in F. It is denoted by $Dom(\Re)$ and can be formally defined as

$$Dom(\Re) = \{x \in E, \exists y \in F : x\Re y\} \subset E.$$

(b). The range (or called the image) of $\Re$ is the set of all elements in F that are related to at least one element in E. It is denoted by $Im(\Re)$ or $ran(\Re)$ and is defined as

$$Im(\Re) = \{y \in F, \exists x \in E : x\Re y\} \subset F.$$

Example 41 *Here are some fundamental examples of relations:*

1. The relations $<, >, \leq, \geq$, (less than, greater than, less than or equal to, greater than or equal to) define binary relations on the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

2. For any set E, the equality relationship on E, denoted by =, is a binary relation on E,

$$\forall x, y \in E : x\Re y \Longleftrightarrow x = y.$$

3. If X is a set, then the inclusion relationship $\subset$ is a binary relation on the power set $E = \mathcal{P}(X)$ of X,

$$\forall A, B \in E : A \mathfrak{R} B \Longleftrightarrow A \subset B.$$

4. The divisibility on $\mathbb{Z}$, denoted by $|$, is a binary relation on $\mathbb{Z}$. This relation is defined as follows

$$\forall x, y \in \mathbb{Z} : x \mathfrak{R} y \Longleftrightarrow x \ \text{divides}\ y \Longleftrightarrow x \mid y.$$

5. Let $E = \mathbb{Z}$ and let $n \in \mathbb{N}^*$ be fixed. The congruence relation modulo n (expressed as "x is congruent to y modulo n") is defined as follows

$$\forall x, y \in \mathbb{Z} : x \mathfrak{R} y \Longleftrightarrow x \equiv y \bmod n \Longleftrightarrow n \ \text{divides}\ (x - y) \ (n \mid (x - y))$$

$$\Longleftrightarrow \exists k \in \mathbb{Z} : x - y = kn.$$

6. In the space E of lines, we define the following binary relations:
(a). The parallelism relation, denoted $\parallel$

$$\forall D_1, D_2 \in E : D_1 \mathfrak{R} D_2 \Longleftrightarrow D_1 \parallel D_2 \ \text{(which is read as:Line } D_1 \text{ is parallel to line } D_2\text{)}$$

(b) The orthogonality relation, denoted $\perp$

$$\forall D_1, D_2 \in E : D_1 \mathfrak{R} D_2 \Longleftrightarrow D_1 \perp D_2 \ \text{(which is read as:Line } D_1 \text{ is orthogonal to } D_2\text{).}$$

Example 42

1. Greater Than Relation. Let $E = \{1, 2, 3\}$ and $F = \{1, 2, 3, 4\}$. Define a relation $\mathfrak{R}$ as "is greater than", then

$$\Gamma = \{(x, y) \in E \times F : x \mathfrak{R} y\} = \{(2, 1), (3, 1), (3, 2)\},$$

we get

$$Dom(\mathfrak{R}) = \{x \in E, \exists y \in F : x \mathfrak{R} y\} = \{2, 3\} \ \text{(since only 2 and 3 have related elements in F).}$$

and

$Im(\mathfrak{R}) = \{y \in F, \exists x \in E : x\mathfrak{R}y\} = \{1, 2\}$ *(since these are the elements in F that are related to elements in E)*

2. Divisibility Relation. Let $E = \{2, 3, 4, 5\}$ *and* $F = \{6, 8, 10, 11\}$. *Define a relation* $\mathfrak{R}$ *as as "divides", then*

$$\Gamma = \{(x, y) \in E \times F : x\mathfrak{R}y\} = \{(2, 6), (2, 8), (2, 10), (3, 6), (4, 8), (5, 10)\},$$

we get

$Dom(\mathfrak{R}) = \{x \in E, \exists y \in F : x\mathfrak{R}y\} = \{2, 3, 4, 5\}$ *(since these numbers in E divide at least one element in F),*

and

$Im(\mathfrak{R}) = \{y \in F, \exists x \in E : x\mathfrak{R}y\} = \{6, 8, 10\}$ *(since these are the elements in F that are related to elements in E).*

Definition 43 ***(Inverse of a binary relation)***
Let $\mathfrak{R}$ *be a binary relation from a set E to a set F.*
1. The inverse relation of $\mathfrak{R}$*, denoted* $\mathfrak{R}^{-1}$*, is the binary relation from F to E defined by*

$$\forall (x, y) \in E \times F : x\mathfrak{R}y \Longleftrightarrow y\mathfrak{R}^{-1}x.$$

This means that y is related to x by $\mathfrak{R}^{-1}$ *if and only if x is related to y by* $\mathfrak{R}$.
2. In other words, the graph of the inverse relation $\mathfrak{R}^{-1}$ *is the set of all reverse-ordred pairs in the relation* $\mathfrak{R}$*. Therefore*

$$\Gamma_{\mathfrak{R}^{-1}} = \{(y, x) \in F \times E : (x, y) \in \Gamma_{\mathfrak{R}}\}.$$

This means that each pair (y, x) *in* $\mathfrak{R}^{-1}$ *corresponds to a pair* (x, y) *in the original relation* $\mathfrak{R}$*, where y is related to x in the inverse relation whenever x is related to y in* $\mathfrak{R}$.

Example 44 *Let us consider the inverses of* ***Greater Than Relation*** *from previous example. Here we have, the inverse relation* $\mathfrak{R}^{-1}$ *is obtained by reversing the pairs in* $\mathfrak{R}$*. So for every pair* $(x, y) \in \Gamma_{\mathfrak{R}}$*, we will have* $(y, x) \in \Gamma_{\mathfrak{R}^{-1}}$*. Thus, the inverse relation* $\mathfrak{R}^{-1}$ *is*

$$\Gamma_{\mathfrak{R}^{-1}} = \{(1, 2), (1, 3), (2, 3)\}.$$

This shows the pairs that are related in the inverse of the ***"greater than"*** *relation, which is equivalent to the* ***"less than"*** *relation in this case.*

Remark 45 *Notice that the inverse relation $\mathfrak{R}^{-1}$ to $\mathfrak{R}$ always exists (because for each pair in $\mathfrak{R}$, we can always create a reversed pair and include it in $\mathfrak{R}^{-1}$).*

Definition 46 *(**Composition of binary relations**)*

Let $\mathfrak{R}$ be a binary relation from set E to set F, and $\mathcal{S}$ be a binary relation from set F to set G.

1. The composition of $\mathfrak{R}$ and $\mathcal{S}$ (sometimes also called the product), denoted by $\mathcal{S} \circ \mathfrak{R}$ (read as S round $\mathfrak{R}$ with $\circ$ representing the small circle used for composition) is the relation from E to G defined as

$$\forall (x, y) \in E \times G : x (\mathcal{S} \circ \mathfrak{R})\, y \Longleftrightarrow \exists z \in F : \begin{cases} x\mathfrak{R}z \\ \text{and} \\ z\mathcal{S}y \end{cases}.$$

In other words: A pair (x, y) is in the graph of the composition $\mathcal{S} \circ \mathfrak{R}$ if and only if there exists an element $z \in F$ such that:

- $(x, z) \in \Gamma_{\mathfrak{R}}$ *(i.e., $x\mathfrak{R}z$).*
- $(z, y) \in \Gamma_{\mathcal{S}}$ *(i.e., $z\mathcal{S}y$).*

2. In the particular case where $E = F = G$, we denote the composition of $\mathfrak{R}$ with itself as $\mathfrak{R}^2 = \mathfrak{R} \circ \mathfrak{R}$.

Remark 47

- *The composition $\mathcal{S} \circ \mathfrak{R}$ means that we first follow the relation $\mathfrak{R}$ from E to F, and then follow the relation $\mathcal{S}$ from F to G.*
- *For every $x \in E$ and $y \in G$, there must be an intermediate element $z \in F$ that connects x to y through the relations $\mathfrak{R}$ and $\mathcal{S}$.*

Example 48 *In an Euclidean affine plane, the composition of the orthogonality relation with itself is indeed parallelism. Specifically, let $E = F = G$ represent the set of lines in an Euclidean affine plane, and let $\mathfrak{R} = \mathcal{S} = \perp$ denote the orthogonality relation. Then,*

$$S \circ \mathfrak{R} = \parallel,$$

is the parallelism relation, because

$$\forall D_1, D_2 \in E : D_1 \parallel D_2 \Longleftrightarrow \exists D_3 \in E : \begin{cases} D_1 \perp D_3 \\ \quad \text{and} \\ D_3 \perp D_2. \end{cases}$$

Thus, we can express this as

$$\perp \circ \perp = \parallel .$$

In geometry, this means that if two lines are both orthogonal to the same line, they are parallel to each other.

Proposition 49 ***(Properties of Composition and inverse of binary Relations)***
Let E, F, G, and H be sets, and let $\mathfrak{R}_1$ (respectively $\mathfrak{R}_2$, and $\mathfrak{R}_3$) be a relation from E to F (respectively from F to G, and from G to H). Then,

1.

$$(\mathfrak{R}_3 \circ \mathfrak{R}_2) \circ \mathfrak{R}_1 = \mathfrak{R}_3 \circ (\mathfrak{R}_2 \circ \mathfrak{R}_1).$$

In other words, the composition of relations is associative.

2. The inverse of the inverse relation is the original relation

$$\left(\mathfrak{R}_1^{-1}\right)^{-1} = \mathfrak{R}.$$

3. The inverse of a composition of relations satisfies

$$(\mathfrak{R}_2 \circ \mathfrak{R}_1)^{-1} = \mathfrak{R}_1^{-1} \circ \mathfrak{R}_2^{-1}.$$

4. In general, the composition of binary relations is not commutative. This means that for two binary relations $\mathfrak{R}$ and $\mathcal{S}$ defined on a set E, we have

$$\mathcal{S} \circ \mathfrak{R} \neq \mathfrak{R} \circ \mathcal{S}.$$

2.9 Properties of binary relations

The commonly encountered binary relations usually have certain special properties. The following definition lists some of the most important properties of binary relations, expressed by formulae of logic.

Definition 50 *Let $\mathfrak{R}$ be a binary relation on a set E. We say that $\mathfrak{R}$ is:*

1. Reflexive if every element of E is related to itself. Formally,

$$\forall x \in E : x\mathfrak{R}x.$$

2. Symmetric if, x is related to y implies that y is also related to x. Formally,

$$\forall x, y \in E : x\mathfrak{R}y \Longrightarrow y\mathfrak{R}x.$$

This means the relation remains unchanged when the order of the two elements in the ordered pair is reversed.

3. Anti-symmetric if, x is related to y and y is related to x, then x must be equal to y. Formally,

$$\forall x, y \in E : \begin{cases} x\mathfrak{R}y \\ \text{and} \\ y\mathfrak{R}x \end{cases} \Longrightarrow x = y.$$

4. Transitive if x is related to y and y is related to z, then x is also related to z. Formally,

$$\forall x, y, z \in E : \begin{cases} x\mathfrak{R}y \\ \text{and} \\ y\mathfrak{R}z \end{cases} \Longrightarrow x\mathfrak{R}z.$$

This means the relation "transfers" through an intermediate element.

Remark 51 *In informal terms, properties (1)–(4) say the following:*

*1. **Reflexive Property:** Every element is related to itself.*

*2. **Symmetric Property:** If any one element is related to any other element, then the second element is related to the first.*

*3. **Anti-symmetric Property:** If any one element is related to a second and that second element is related to the first, then the two elements must be identical.*

*4. **Transitive Property:** If any one element is related to a second and that second element is related to a third, then the first element is related to the third.*

Example 52 ***(Basic Examples)***

1. The relation defined by equality $=$ *on* $\mathbb{R}$*, is reflexive, symmetric, transitive, and anti-symmetric.*

2. The inclusion relation $\subset$ *in the set* $\mathcal{P}(X)$ *of subsets of* X *is reflexive, anti-symmetric, and transitive, but not symmetric (e.g.,* $A = \{1\}, B = \{1, 2\}$*).*

3. The divisibility relation $|$ *on the set* $\mathbb{N}^*$ *is reflexive, anti-symmetric, and transitive, but not symmetric (e.g.,* $x = 2, y = 4$*). Furthermore, the relation is reflexive and transitive but not symmetric (e.g.,*$x = 2, y = 4$*), and it is not anti-symmetric on* $\mathbb{Z}^*$ *(e.g.,* $x = 2, y = -2$*).*

4. For any $n \in \mathbb{N}^*$*, the congruence relation modulo* n *on* $\mathbb{Z}$*, is reflexive, symmetric, and transitive, but not anti-symmetric (e.g.,* $n = 2$*,* $x = 8, y = -8$*).*

Remark 53 *Now consider what it means for a relation not to have one of the properties defined previously. Recall that the negation of a universal assertion is existential. Hence if* $\mathfrak{R}$ *is a relation on a set E, then*

1. ***Non-reflexivity:*** $\mathfrak{R}$ *is not reflexive if and only if there is an element x in E such that* $x\mathfrak{R}x$ *(that is, such that* $(x, x) \in \Gamma_{\mathfrak{R}}$*).*

$$\exists x \in E : x\mathfrak{R}x.$$

2. ***Non-symmetry:*** $\mathfrak{R}$ *is not symmetric if and only if there are elements x and y in E such that* $x\mathfrak{R}y$ *but* $y\mathfrak{R}x$ *(that is, such that* $(x, y) \in \Gamma_{\mathfrak{R}}$ *but* $(y, x) \notin \Gamma_{\mathfrak{R}}$*).*

$$\exists x, y \in E : x\mathfrak{R}y \wedge y\mathfrak{R}x.$$

3. ***Non-antisymmetry:*** $\mathfrak{R}$ *is not Anti-symmetric if and only if there are elements x and y in E such that* $x\mathfrak{R}y$ *and* $y\mathfrak{R}x$ *but* $x \neq y$ *(that is, such that* $(x, y) \in \Gamma_{\mathfrak{R}}$ *and* $(y, x) \in \Gamma_{\mathfrak{R}}$ *but* $x \neq y$*).*

$$\exists x, y \in E : (x\mathfrak{R}y \wedge y\mathfrak{R}x) \wedge x \neq y.$$

4. ***Non-transitivity:*** $\mathfrak{R}$ *is not transitive if and only if there are elements x, y and z in E such that* $x\mathfrak{R}y$ *and* $y\mathfrak{R}z$ *but* $x\mathfrak{R}z$ *(that is, such that* $(x, y) \in \Gamma_{\mathfrak{R}}$ *and* $(y, z) \in \Gamma_{\mathfrak{R}}$ *but* $(x, z) \notin \Gamma_{\mathfrak{R}}$*).*

$$\exists x, y \in E : (x\mathfrak{R}y \wedge y\mathfrak{R}x) \wedge x\mathfrak{R}z.$$

2.10 Equivalence Relations, Equivalence Classes and Partitions

In this section we will take a closer look at relations of equivalence and give a precise mathematical definition of the notion of equivalence relation; we will see how an equivalence relation divides (or partitions) the set on which it is defined into equivalence classes and how, consequently, equivalence relations and partitions are two sides of the same coin.

2.10.1 Equivalence Relations

Definition 54 ***(Equivalence Relation)*** *Let E be a set and $\mathfrak{R}$ a binary relation on E.*

1. We call $\mathfrak{R}$ an equivalence relation on E if it is reflexive, symmetric and transitive.

2. If $\mathfrak{R}$ is an equivalence relation on a set E and $x, y \in E$ with $x\mathfrak{R}y$, then we say that x is equivalent to y under the equivalence relation $\mathfrak{R}$, or that x and y are equivalent with respect to $\mathfrak{R}$, or simply that x and y are equivalent modulo $\mathfrak{R}$.

Example 55 *The following are equivalence relations on the set E:*

1. The equality relation in any set E is an equivalence relation.

2. For any $n \in \mathbb{N}^*$, the congruence relation modulo n on $\mathbb{Z}$ is an equivalence relation.

3. The relation of "having the same absolute value" on $\mathbb{R}$ is an equivalence relation,

$$\forall x, y \in \mathbb{R} : x\mathfrak{R}y \Longleftrightarrow |x| = \left|y\right|.$$

4. The relation "being the same age" in the set of people E is an equivalence relation,

$$\forall x, y \in E : x\mathfrak{R}y \Longleftrightarrow age(x) = age(y),$$

5. In the set of all lines in the plane, the relation " being parallel" is an equivalence relation.

2.10.2 Equivalence class and quotient set

Definition 56 *Let $\mathfrak{R}$ be an equivalence relation on a set E.*

1. For each $x \in E$, the equivalence class of x (modulo $\mathfrak{R}$), denoted by $\dot{x}$ (or $\bar{x}$, or $\hat{x}$, or $cl(x)$), is the set of all elements in E that are related to x under the relation $\mathfrak{R}$. That is,

$$\dot{x} = \{y \in E : y\mathfrak{R}x\} \subset E.$$

In other words, the equivalence class of x consists of all elements that are equivalent to x according to the equivalence relation $\mathfrak{R}$, meaning

$$y \in \dot{x} \Longleftrightarrow x\mathfrak{R}y.$$

2. Any element $t \in \dot{x}$ that belongs to the equivalence class $\dot{x}$ is called a representative of the class $\dot{x}$ (An equivalence class can be represented by any element in that equivalence class).

3. The quotient set of E by $\mathfrak{R}$ is the set of all equivalence classes of elements in E under $\mathfrak{R}$. It is denoted by $E/\mathfrak{R}$. Thus, we have

$$E/\mathfrak{R} = \{\dot{x} : x \in E\} \subset \mathcal{P}(E).$$

where $\dot{x}$ represents the equivalence class of x under the relation $\mathfrak{R}$.

Example 57 *Consider the equivalence relation " the congruence relation modulo 3 " on the set of integers $\mathbb{Z}$, defined by*

$$\forall x, y \in \mathbb{Z} : x\mathfrak{R}y \Longleftrightarrow \exists k \in \mathbb{Z} : x - y = 3k.$$

Then we have, the equivalence class of $x \in \mathbb{Z}$ is

$$\dot{x} = \{y \in \mathbb{Z} : y\mathfrak{R}x\} = \{y \in \mathbb{Z} : y = x + 3k, k \in \mathbb{Z}\} = x + 3\mathbb{Z}.$$

We get

$$\dot{0} = \{y = 3k, k \in \mathbb{Z}\} = 3\mathbb{Z}, \dot{1} = \{y = 3k + 1, k \in \mathbb{Z}\} = 3\mathbb{Z}+1, \dot{2} = \{y = 3k + 2, k \in \mathbb{Z}\} = 3\mathbb{Z}+2.$$

The quotient set of is given by

$$\mathbb{Z}/\mathfrak{R} = \left\{\dot{0}, \dot{1}, \dot{2}\right\} \overset{denoted}{=} \mathbb{Z}/3\mathbb{Z}.$$

This quotient set consists of the three distinct equivalence classes of integers modulo 3*. Each equivalence class corresponds to the remainder when dividing by* 3*.*

Proposition 58 ***(Basic Properties of Equivalence Classes)***
Let $\mathfrak{R}$ be an equivalence relation on a set E. The equivalence classes have the following fundamental properties:

1. Each equivalence class is nonempty,

$$\forall x \in E : \dot{x} \neq \phi.$$

2. Any two equivalence classes are either identical or disjoint,

$$\textit{(i)}\ \forall x, y \in E : x\mathfrak{R}y \Longleftrightarrow \dot{x} = \dot{y},$$

and

$$\textit{(ii)}\ \forall x, y \in E : x\mathfrak{R}y \Longleftrightarrow \dot{x} \cap \dot{y} = \phi.$$

3. The equivalence classes cover E,

$$\bigcup_{x\in E}\dot{x} = E.$$

Corollary 59 ***(Partition Associated with an Equivalence Relation)***

If $\mathfrak{R}$ is an equivalence relation on a set E, then the quotient set

$$E/\mathfrak{R} = \{\dot{x} : x \in E\}$$

is a partition of E (the collection of distinct equivalence classes forms a partition of E, meaning that E is divided into non-overlapping, mutually exclusive subsets).

Proof. This result follows immediately from the previous proposition (58). It remains to verify that all three conditions of the partition definition are satisfied.

■

2.11 Orders Relations

An order relation (or ordering) on a set E is a type of binary relation that describes a way to compare elements in E in terms of "order." Specifically, an order relation $\mathfrak{R}$ on a set E is a binary relation that satisfies certain properties.

Definition 60

1. A binary relation $\mathfrak{R}$ on a set E is an order relation if and only if it is reflexive, antisymmetric, and transitive. In other words, a binary relation $\mathfrak{R}$ is an order relation if it satisfies the following three properties:

• *Reflexivity:*

$$\forall x \in E : x\mathfrak{R}x.$$

• *Antisymmetry:*

$$\forall x, y \in E : \begin{cases} x\mathfrak{R}y \\ and \\ y\mathfrak{R}x \end{cases} \Longrightarrow x = y.$$

• *Transitivity:*

$$\forall x, y, z \in E : \begin{cases} x\mathfrak{R}y \\ and \\ y\mathfrak{R}z \end{cases} \Longrightarrow x\mathfrak{R}z.$$

2. *A set E equipped with an order relation $\mathfrak{R}$ is called an ordered set, denoted by $(E, \mathfrak{R})$.*

3. *An order relation is frequently denoted by $\leq$ instead of $\mathfrak{R}$, and $x \leq y$ is read as "x is less than or equal to y" or "y is greater than or equal to x."*

Example 61

• *The usual relations of equality, superiority, and inferiority $(=, \geq, \leq)$ are order relations on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$.*

• *The subset relation $\subset$ on the power set of a set E is an order relation.*

• *The divisibility relation on the set $\mathbb{N}^*$ is an order relation.*

• *The relation $\mathfrak{R}$ defined by*

$$\forall x, y \in \mathbb{N}^* : x\mathfrak{R}y \Longleftrightarrow \exists n \in \mathbb{N}^* : y = x^n,$$

is an order relation on $\mathbb{N}^*$. This relation can also be expressed as "y is a nonzero integer power of x."

• $(\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \leq), (\mathbb{N}^*, |), (\mathcal{P}(S), \subset)$ *are ordered sets.*

2.11.1 Total Order and Partial Order Relations

There are two main types of order relations: Total order and Partial order. In a total order relation, every pair of elements in the set is comparable; in a partial order relation, not every pair of elements needs to be comparable. A set with a total order is called a totally ordered set, and a set with a partial order is called a partially ordered set.

Definition 62 *Let $\mathfrak{R}$ be an order relation on a set E. Then*

1. Two elements x and y of E are said to be comparable with respect to $\mathfrak{R}$ if and only if

$$x\mathfrak{R}y \text{ or } y\mathfrak{R}x.$$

Otherwise, x and y are called noncomparable, which means

$$x\mathfrak{R}y \text{ et } y\mathfrak{R}x.$$

2. If every two elements of E are comparable, then $\mathfrak{R}$ is a total order on E (or that E is totally ordered by $\mathfrak{R}$). Otherwise, we say that $\mathfrak{R}$ is a partial order (or that E is partially ordered), which means there are at least two elements noncompatible with respect to $\mathfrak{R}$.

Example 63

1. The "less than or equal to" relation $\leq$ on the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ is a total order relation.

2. The subset relation $\subset$ on the power set of a set E is a partial order relation because, some sets are not comparable (e.g., $X_1 = \{5\}$ and $X_2 = \{0\}$ are not comparable under $\subset$,

$$X_1 \not\subset X_2 \text{ and } X_2 \not\subset X_1.$$

3. The divisibility relation $|$ on the set $\mathbb{N}^*$ is a partial order because for example, 3 *and* 4 *are not comparable,*

$$3 \nmid 4 \text{ and } 4 \nmid 3.$$

2.11.2 Special elements of ordered sets

In an ordered set, there are several special elements that play significant roles in its structure and properties. Some of these special elements include: Upper Bound, Lower Bound, Greatest Element (Maximum), Least Element (Minimum), Greatest Lower Bound (Infimum), and Least Upper Bound (Supremum).

2.11.2.1 Upper and Lower Bounds

Definition 64 *Let $(E, \preceq)$ be an ordered set and A be a non-empty subset of E.*
1. An element $l \in E$ is called a lower bound (or minorant) of A if l is less than or equal to every element of A (or if every element of A is related to l in the reverse order). In other words,

$$\forall a \in A : l \preceq a \text{ (or } \forall a \in A : l\mathfrak{R}a).$$

2. An element $u \in E$ is called an upper bound (or majorant) of A if every element of A is less than or equal to u (or if every element of A is related to u in the order). In other words,

$$\forall a \in A : a \preceq u \text{ (or } \forall a \in A : a\mathfrak{R}u).$$

3. The set A is said to be bounded below or minorized (respectively: bounded above or majorized) in E if A has at least one lower bound (respectively: one upper bound) in E.
4. The set A is called bounded in E if A is both bounded below and bounded above in E.
5. If A is bounded above (respectively: bounded below), then we denote by $\mathcal{UB}(A, E)$ (respectively: $\mathcal{LB}(A, E)$) the set of all upper bounds (respectively: lower bounds) of A in

E. Then

$$\mathcal{UB}(A, E) = \{u \in E, \forall a \in A : a \preceq u\},$$

and

$$\mathcal{LB}(A, E) = \{l \in E, \forall a \in A : l \preceq a\}.$$

2.11.2.2 Greatest element and Least element

Definition 65 *Let $(E, \preceq)$ be an ordered set and A be a non-empty subset of E.*

1. An element $\alpha \in E$ is called the greatest element (or maximum) of A, denoted $\max(A)$, if and only if α is an upper bound of A and $\alpha \in A$. That is,

$$\alpha = \max(A) \Longleftrightarrow \begin{cases} \alpha \in A, \\ \text{and} \\ \forall a \in A : a \preceq \alpha. \end{cases}$$

In other words: An element α is called the greatest (or maximum) element of a set A if it is greater than or equal to every other element in A.

2. An element $\beta \in E$ is called the least element (or minimum) of A, denoted $\min(A)$, if β is a lower bound of A and belongs to A. That is,

$$\beta = \min(A) \Longleftrightarrow \begin{cases} \beta \in A, \\ \text{and} \\ \forall a \in A : \beta \preceq a. \end{cases}$$

In other words, an element β is called the least (or minimum) element of a set A if it is less than or equal to every other element in A.

Example 66

1. If $(E, \preceq) = (\mathbb{R}, \leq)$ (with the usual order relation), then:

• *If* $A = [0,1]$, *then* $\max(A) = 1$ *and* $\min(A) = 0$.

• *If* $]0,1]$, *then A has no minimum, but its maximum is* 1.

• *If* $]0,1[$, *then A has neither a maximum nor a minimum.*

• *If* $A =]-\infty,1]$, *then A is bounded above but not below (i.e., unbounded below). The set of its upper bounds is* $[1,+\infty[$.

• *If* $A = [-2,+\infty[$, *then A is bounded below but not above (i.e., unbounded above). The set of its lower bounds is* $]-\infty,-2]$. *The minimum of A is* -2.

2. If $(E,\preceq) = (E,\subset)$, *then the greatest element of* $\mathcal{P}(E)$ *is E itself and the least element is the empty set* ϕ. *So*

$$\max(\mathcal{P}(E)) = E \text{ and } \min(\mathcal{P}(E)) = \phi.$$

3. If $(E,\preceq) = (\mathbb{N}^*,|)$ *(ordered by the divisibility relation* $|$*) and* $A = \{1,2,3\}$, *then (the minimum)* $\min(A) = 1$, *but A does not have a maximum (Because* $2 \nmid 3$, *i.e,* 2 *and* 3 *non-comparable in terms of divisibility).*

2.11.2.3 Greatest Lower Bound and Least Upper Bound

Definition 67 *Let* $(E,\preceq)$ *be an ordered set and A be a non-empty subset of E.*

1. If there exists a least element in the set of upper bounds of A, this element is called the least upper bound ***(abbreviated as LUB)*** *or supremum of A, denoted by* $\sup(A)$ *or* $\sup\limits_E(A)$. *In other words,*

$$\sup_E(A) = \min\left(\mathcal{UB}(A,E)\right).$$

where $\mathcal{UB}(A,E)$ *is the set of all upper bounds of A in E.*

2. If there exists a greatest element among the lower bounds of A, it is called the greatest lower bound (abbreviated as ***GLB****) or infimum of A, denoted by* $\inf(A)$ *or* $\inf\limits_E(A)$. *In other words,*

$$\inf_E(A) = \max\left(\mathcal{LB}(A,E)\right).$$

where $\mathcal{LB}(A,E)$ is the set of all lower bounds of A in E.

Remark 68 *A set A does not necessarily have a minimum or maximum, similarly does it necessarily have an infimum or supremum. Moreover, the infimum or supremum of A, when they exist, are not necessarily elements of A.*

Example 69

1. If $(E,\preceq) = (\mathbb{R},\leq)$ (with the usual order relation) and $A =]0,1[$ (the open interval between 0 and 1). Then

• The infimum of A is 0, The supremum is 1 (Note that the infimum and supremum are not necessarily elements of the set A)

2. If $(E,\preceq) = (E,\subset)$, then

$$\sup(\mathcal{P}(E)) = E \text{ and } \inf(\mathcal{P}(E)) = \phi.$$

Proposition 70 ***(Properties)*** *Let $(E,\preceq)$ be an ordered set and A be a non-empty subset of E.*

*1. **Uniqueness of Greatest and Least Elements**: Both the greatest and least elements are unique when they exist. This means that there can be only one greatest element and one least element in the set.*

*2. **Uniqueness of Supremum and Infimum:** If a set has a supremum or an infimum, it is unique. However, the converse is not true.*

*3. **Greatest Element and Supremum:** If a set has a greatest element, that element is the supremum. However, the converse is not true (a set may have a supremum without having a greatest element. The supremum is the **Least** element among all the upper bounds of the set, but it does not need to be an element of the set itself).*

*4. **Least Element and infimum:** If a set has a least element, that element is the infimum. However, the converse is not true (a set may have an infimum without having* *a least element. The infimum is the greatest element among all the lower bounds of the set, but it does not need to be an element of the set itself).*