Exercises on Set Theory

Exercise 1 (Set Operations)**.** Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$, and $C = \{2, 4, 6, 8\}$. Compute:

(a) $A \cup B$

(b) $A \cap C$

(c) $B \setminus A$

(d) $A \triangle C$ (symmetric difference)

(e) $\mathcal{P}(A \cap B)$ (power set of $A \cap B$)

Solution 1. (a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

(b) $A \cap C = \{2, 4\}$

(c) $B \setminus A = \{6, 7\}$

(d) $A \triangle C = (A \setminus C) \cup (C \setminus A) = \{1, 3, 5\} \cup \{6, 8\} = \{1, 3, 5, 6, 8\}$

(e) $A \cap B = \{3, 4, 5\}$, so $\mathcal{P}(A \cap B) = \{\emptyset, \{3\}, \{4\}, \{5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}\}$

Exercise 2 (Cartesian Product and Ordered Pairs)**.** Using the set-theoretic definition of ordered pairs $(x, y) = \{\{x\}, \{x, y\}\}$:

(a) Write explicitly the ordered pair $(1, 2)$

(b) Prove that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$

Solution 2. (a) $(1, 2) = \{\{1\}, \{1, 2\}\}$

(b) **Proof:**
If $a = c$ and $b = d$, then clearly $(a, b) = (c, d)$.
Now assume $(a, b) = (c, d)$, i.e., $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$.
We consider two cases:

- If $a = b$, then $\{\{a\}, \{a, b\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}$. So $\{\{c\}, \{c, d\}\} = \{\{a\}\}$, which implies $\{c\} = \{c, d\} = \{a\}$, so $c = d = a = b$.
- If $a \neq b$, then $\{a\} \neq \{a, b\}$. Since $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$, we must have $\{a\} = \{c\}$ (which gives $a = c$) and $\{a, b\} = \{c, d\}$ (which with $a = c$ gives $b = d$).

In both cases, $a = c$ and $b = d$.

Exercise 3 (Relations and Their Properties)**.** Let R be a relation on $\mathbb{Z}$ defined by xRy if and only if $x - y$ is even. Determine whether R is:

(a) Reflexive

(b) Symmetric

(c) Transitive

(d) Antisymmetric

Justify your answers.

Solution 3. (a) **Reflexive:** Yes. For any $x \in \mathbb{Z}$, $x - x = 0$ which is even, so xRx.

(b) **Symmetric:** Yes. If xRy, then $x - y$ is even, so $y - x = -(x - y)$ is also even, hence yRx.

(c) **Transitive:** Yes. If xRy and yRz, then $x - y$ and $y - z$ are even. Their sum $(x - y) + (y - z) = x - z$ is even, so xRz.

(d) **Antisymmetric:** No. Counterexample: $2R4$ and $4R2$ (since $2 - 4 = -2$ and $4 - 2 = 2$ are both even), but $2 \neq 4$.

Exercise 4 (Functions and Their Properties)**.** For each function below, determine whether it is injective, surjective, bijective, or none:

(a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$

(b) $g : \mathbb{Z} \to \mathbb{Z}, g(n) = 2n$

(c) $h : \mathbb{R} \to [0, \infty), h(x) = x^2$

(d) $k : \mathbb{N} \to \mathbb{N}, k(n) = n + 1$

Solution 4. (a) $f(x) = x^3$ is bijective (both injective and surjective) on $\mathbb{R}$.

(b) $g(n) = 2n$ is injective but not surjective (odd numbers are not in the image).

(c) $h(x) = x^2$ is neither injective (e.g., $h(1) = h(-1)$) nor surjective (negative numbers are not in the image, though the codomain is $[0, \infty)$, so actually it is surjective onto its codomain but not injective).

(d) $k(n) = n + 1$ is injective but not surjective (1 is not in the image).

Exercise 5 (Russell's Paradox)**.** Explain in your own words why the "set of all sets that do not contain themselves" leads to a paradox.

Solution 5. Let $R = \{A \mid A \notin A\}$ be the "set of all sets that do not contain themselves".
If $R \in R$, then by definition R must satisfy $R \notin R$.
If $R \notin R$, then R satisfies the condition for belonging to R, so $R \in R$.
Both assumptions lead to a contradiction, showing that such a set R cannot exist in consistent set theory.

Exercise 6 (Zermelo-Fraenkel Axioms)**.** Which ZF axiom justifies each of the following?

(a) The existence of the set $\{a, b\}$ for any two sets a and b

(b) The existence of the union $\bigcup_{i \in I} A_i$ for any family of sets $\{A_i\}_{i \in I}$

(c) The existence of the power set $\mathcal{P}(A)$ for any set A

(d) The statement that two sets with the same elements are equal

Solution 6. (a) Axiom of Pairing

(b) Axiom of Union

(c) Axiom of Power Set

(d) Axiom of Extensionality

Exercise 7 (Cardinality and Equipollence)**.** (a) Prove that the interval $(0, 1)$ has the same cardinality as $\mathbb{R}$.

(b) Show that the set of integers $\mathbb{Z}$ is countable.

(c) What is the cardinality of $\mathcal{P}(\mathbb{N})$? Justify your answer.

Solution 7. (a) The function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$ is a bijection, so $(0, 1) \sim \mathbb{R}$.

(b) We can list all integers: $0, 1, -1, 2, -2, 3, -3, \ldots$, which gives a bijection with $\mathbb{N}$. Alternatively, the function $f : \mathbb{Z} \to \mathbb{N}$ defined by:

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n - 1 & \text{if } n < 0 \end{cases}$$

is a bijection.

(c) By Cantor's Theorem, $\text{card}(\mathbb{N}) < \text{card}(\mathcal{P}(\mathbb{N}))$. The cardinality of $\mathcal{P}(\mathbb{N})$ is $2^{\aleph_0}$, which is the cardinality of the continuum.

Exercise 8 (Axiom of Choice)**.** Explain what the Axiom of Choice says and give an example of a situation where it is needed.

Solution 8. The Axiom of Choice states that for any collection of non-empty sets, there exists a function (called a choice function) that selects exactly one element from each set in the collection.
Example: Consider the collection of all non-empty subsets of $\mathbb{R}$. The Axiom of Choice guarantees that there exists a function that assigns to each non-empty subset of $\mathbb{R}$ one of its elements. This is needed to prove many important results in mathematics, such as:

- Every vector space has a basis
- The product of any collection of non-empty sets is non-empty
- Zorn's Lemma and the Well-Ordering Theorem