

Chapter 01: Properties of the Real Numbers

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Institute of Science and Technology

1st Year Common Core ST (LMD) And 1st Year Engineering

Module: Analysis 1

Semester 1

Academic Year: 2025/2026

- References
- Sets
- Operations with real numbers
- Axiomatic definition of real numbers
- Archimedean property
- Absolute value
- The floor function (Integer part)
- Intervals in $\mathbb{R}$
- More About: Lower bounds, upper bounds, infimum, supremum, Maximum, Minimum
- Characterization of the supremum and of the infimum
- The density of $\mathbb{Q}$ in $\mathbb{R}$

بالعربية:

- **بابا حامد، بن حبيب**، التحيل 1 تذكير بالدروس و تمارين محلولة عدد 300 ترجمة الحفيظ مقران، ديوان المطبوعات الجامعية (الفصل الأول) .

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- **Terence Tao**, Analysis 1 (3rd edition), Springer (2016).

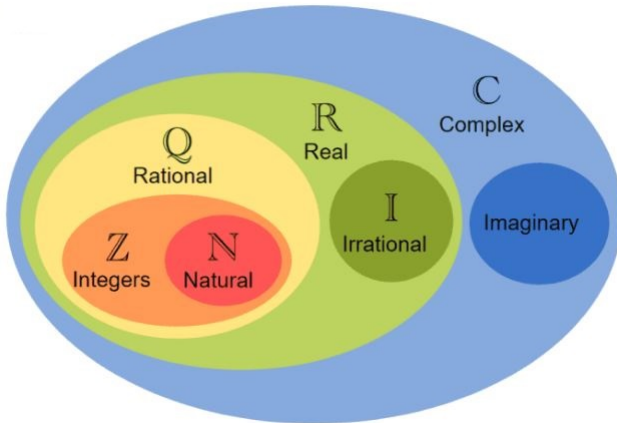
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Sets:

- In mathematics, the set is a fundamental concept, which refers to class or collection of objects with specified characteristics.
- For example, we can refer to the set of university students, or the set of letters in the English alphabet.
- The individual objects within a set are known as members or elements.
- Any portion of a set is called a subset of the original set, for example, $\{A, B, C\}$ is a subset of the English alphabet set.
- A set that contains no elements is referred to as the empty set or null set $\emptyset$.

Numbers:



Source:

<https://owlcation.com/stem/How-to-use-complex-numbers>

Real Numbers:

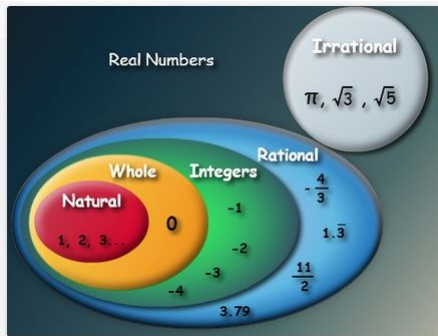


Figure: :

<https://empoweryourknowledgeandhappytrivia.wordpress.com/2016/09/06/what-are-the-types-of-real-numbers/>

There are different types of numbers already familiar we students:

- Natural numbers: $\{1, 2, 3, \dots\}$, also known as positive integers, are used for counting the members of a set.
- The sum $a + b$ and product $a \cdot b$ or ab of any two numbers a and b is also a natural number. This is often expressed by stating that the set of natural numbers is closed under the operations of addition and multiplication, or satisfies the closure property with respect to these operators.
- Negative integers and zero, denoted by $\{-1, -2, -3, \dots\}$ and 0 respectively, were introduced to allow for solutions of equation such as $x + b = a$ where a and b are any natural numbers. This leads to the operation of subtraction, the inverse of addition, and we write $x = a - b$.
- The set of positive and negative integers and zero is called the set of integers.

- Rational numbers or fractions such as $\frac{4}{3}$ arose to permit solutions of equation such as $bx = a$ for all integers a and b where $a \neq 0$. This leads to the operation of division; the inverse of multiplication, and we write $x = \frac{a}{b}$ or $a \div b$, where a is the numerator and b is the denominator. The set of integers is a subset of the rational numbers, since integers correspond to rational numbers where $b = 1$.
- Irrational numbers such as π and $\sqrt{2}$ are numbers which are not rational, meaning they cannot be expressed as $\frac{a}{b}$ where a and b are integers and $b \neq 0$.
- The set of rational and irrational numbers is called the set of real numbers.

Operations with Real numbers

When considering numbers a , b , and c from the set of real numbers $\mathbb{R}$, we have the following operations and properties:

- ① **Closure under Addition:** $a + b$ is a real number.
- ② **Commutative Law of Addition:** $a + b = b + a$.
- ③ **Associative Law of Addition:** $a + (b + c) = (a + b) + c$.
- ④ **Additive Identity, call it 0:** $a + 0 = 0 + a = a$.
- ⑤ **Addition inverse:** For any a there exists x such that $x + a = 0$.
(denoted by $-a$).
- ⑥ **Closure under Multiplication:** $a \cdot b$ is a real number.
- ⑦ **Commutation Law of Multiplication:** $a \cdot b = b \cdot a$.
- ⑧ **Associative Law of Multiplication:** $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- ⑨ **Multiplicative Identity, call it 1:** $1 \cdot a = a \cdot 1 = a$.
- ⑩ **Multiplicative Inverse:** For $a \neq 0$: for every a , there exists x such that $a \cdot x = 1$ (denoted by $\frac{1}{a}$ or a^{-1}).
- ⑪ **Distributive Law of Multiplication over addition:**
 $a \cdot (b + c) = a \cdot b + a \cdot c$.

- Generally, any set; like $\mathbb{R}$, whose elements satisfy the above properties is known as a field.
- For example, the set of integers $\mathbb{Z}$ is not a field because it does not satisfy the property of multiplicative inverse (10).
- The field has not the same meaning of the word field, which is usually used in physics.

Axiomatic definition of real numbers

- The set of **real numbers** is denoted by $\mathbb{R}$, on which two internal composition laws are define:
 - **addition**

$$\begin{aligned} " + " : \mathbb{R} \times \mathbb{R} &\rightarrow \mathbb{R} \\ (x, y) &\rightarrow x + y \end{aligned}$$

- **multiplication**

$$\begin{aligned} " \cdot " : \mathbb{R} \times \mathbb{R} &\rightarrow \mathbb{R} \\ (x, y) &\rightarrow x \cdot y \end{aligned}$$

such that $(\mathbb{R}, +, \cdot)$ forms a **complete ordered field**.

Ordered Axioms: Complete Summary

The Four Axioms

For all $a, b, c \in \mathbb{R}$:

① **Trichotomy:**

$$a < b, a = b, \text{ or } b < a$$

② **Transitivity:**

$$a < b \text{ and } b < c \Rightarrow a < c$$

③ **Additive:**

$$a < b \Rightarrow a + c < b + c$$

④ **Multiplicative:**

$$a < b \text{ and } c > 0 \Rightarrow ac < bc$$

Key Consequence

Negative Multiplication:

$$a < b \text{ and } c < 0 \Rightarrow ac > bc$$

Examples:

$$2 < 5 \Rightarrow 2 + 3 < 5 + 3$$

$$2 < 5 \Rightarrow 2(-2) > 5(-2)$$

Proof: $a < b$ and $c < 0 \Rightarrow ac > bc$

- Given: $a < b$ and $c < 0$
- Since $c < 0$, then $0 < -c$ (Add $-c$ to both sides)
- Apply (4): $a < b$ and $0 < -c \Rightarrow a(-c) < b(-c)$
- Simplify: $-ac < -bc$
- Add $ac + bc$ to both sides: $bc < ac$
- Therefore: $ac > bc$

Completeness Axiom: No Holes in $\mathbb{R}$

The Axiom Every non-empty set $S \subseteq \mathbb{R}$ that is bounded above has a least upper bound ($\sup S$) in $\mathbb{R}$.

Definitions

- Upper bound: M such that $x \leq M$ for all $x \in S$
- Supremum: Least such M

Example: $S = \{1, 2, 3\}$

- **Upper bounds:** $3, 4, 5, \dots$
- **Supremum:** $\sup S = 3$

Why It Matters

- $\{x \in \mathbb{Q} : x^2 < 2\}$ has no sup in $\mathbb{Q}$
- But $\sup = \sqrt{2}$ exists in $\mathbb{R}$
- Distinguishes $\mathbb{R}$ from $\mathbb{Q}$

Consequence Real number line is continuous - no missing points
Essential for calculus and continuous functions

Proof: $\sqrt{2}$ Exists via Completeness

Setup

$$E = \{x \geq 0 : x^2 < 2\}, \quad r = \sup E$$

Case 1: $r^2 < 2$ leads to contradiction

- Find ε such that $(r + \varepsilon)^2 < 2$
- Then $r + \varepsilon \in E$ but $r + \varepsilon > r$
- Contradicts r being upper bound

Case 2: $r^2 > 2$ leads to contradiction

- Find ε such that $(r - \varepsilon)^2 > 2$
- Then $r - \varepsilon$ is smaller upper bound
- Contradicts r being least upper bound

Conclusion: $r^2 = 2$, so $\sqrt{2} = r$ exists

Proof that $\sqrt{2}$ Exists in $\mathbb{R}$

Goal: Show there exists $r \in \mathbb{R}$ such that $r^2 = 2$

Strategy: Use the Completeness Axiom

Step 1: Define the Set

$$E = \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 < 2\}$$

Step 2: Show E has Supremum

- E is non-empty ($1 \in E$ since $1^2 = 1 < 2$)
- E is bounded above (by 2, since $x < 2 \Rightarrow x^2 < 2 < 4$)
- By Completeness Axiom: $r = \sup E$ exists
- Since $1 \in E$ and 2 is upper bound: $1 \leq r \leq 2$

Proof Continued: $r^2 = 2$ by Contradiction

Case 1: Suppose $r^2 < 2$ so $(2 - r^2) > 0$

- Choose small ε with $0 < \varepsilon < 1$
- $(r + \varepsilon)^2 = r^2 + 2\varepsilon r + \varepsilon^2$
- Since $r \leq 2$ and $\varepsilon^2 \leq \varepsilon$: $(r + \varepsilon)^2 \leq r^2 + 5\varepsilon$
- Choose ε so that $r^2 + 5\varepsilon < 2$ then $0 < \varepsilon < \frac{2-r^2}{5}$
- Then $(r + \varepsilon)^2 < 2$, so $r + \varepsilon \in E$
- **Contradiction:** $r + \varepsilon > r$ but r is supposed to be upper bound

Case 2: Suppose $r^2 > 2$ so $(r^2 - 2) > 0$

- Choose small ε with $0 < \varepsilon < 1$
- $(r - \varepsilon)^2 = r^2 - 2\varepsilon r + \varepsilon^2$
- Since $r \leq 2$: $(r - \varepsilon)^2 \geq r^2 - 4\varepsilon$
- Choose ε so that $r^2 - 4\varepsilon > 2$ then $0 < \varepsilon < \frac{r^2-2}{4}$
- Then $(r - \varepsilon)^2 > 2$, so $r - \varepsilon$ is also an upper bound for E
- **Contradiction:** $r - \varepsilon < r$ but r is supposed to be least upper bound

Conclusion and Significance

Conclusion

- Both $r^2 < 2$ and $r^2 > 2$ lead to contradictions
- Therefore, $r^2 = 2$
- $\sqrt{2} = r = \sup E$ exists in $\mathbb{R}$

What This Shows

- Completeness Axiom fills the "gaps" in $\mathbb{Q}$
- $\sqrt{2}$ exists in $\mathbb{R}$ but not in $\mathbb{Q}$
- Method generalizes to prove existence of any $\sqrt{a}$

Why It Matters

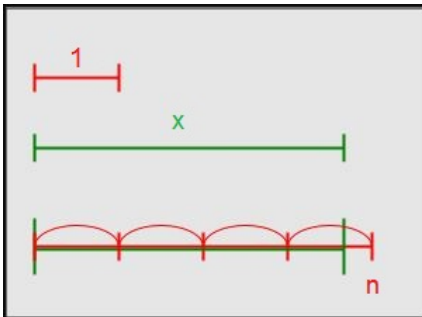
- Distinguishes $\mathbb{R}$ from $\mathbb{Q}$
- Enables calculus and continuous functions
- Foundation for real analysis

Completeness Axiom $\Rightarrow$ No holes in real number line

Archimedean Property

- The field of real numbers $\mathbb{R}$ satisfies **the Archimedean principle**, which is stated as follows:

$$\forall x \in \mathbb{R}^+ : \exists n \in \mathbb{N} : x < n.$$



- This means that $\mathbb{N}$ is **not bounded from above**.

Proof: $\mathbb{N}$ is Unbounded Above

Step 1: Assume the opposite Assume $\mathbb{N}$ is bounded above in $\mathbb{R}$

Step 2: Apply Completeness Since $\mathbb{N}$ is non-empty and bounded above, by Completeness Axiom:

$$\exists M = \sup \mathbb{N}$$

Step 3: Find contradiction Since M is the least upper bound, $M - 1$ is not an upper bound.

$$\exists n \in \mathbb{N} \text{ such that } n > M - 1$$

But then $n + 1 \in \mathbb{N}$ and $n + 1 > M$

Step 4: Conclusion Contradiction: M cannot be an upper bound if $n + 1 > M$

Therefore, $\mathbb{N}$ is unbounded above

Absolute Value:

- **Definition:** The absolute value is a mapping from $\mathbb{R}$ to the set of positive real numbers $\mathbb{R}^+$, denoted by $|\cdot|$ and defined by: $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}^+$

$$|\cdot| : \mathbb{R} \rightarrow \mathbb{R}^+$$

$$x \mapsto |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

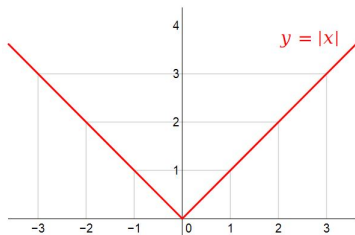


Figure: Source: <https://upload.wikimedia.org>

- **Properties:**

- ① $|x| \geq 0, \forall x \in \mathbb{R}.$
- ② $|x| = 0 \Leftrightarrow x = 0.$
- ③ $-|x| \leq x \leq |x|; \forall x \in \mathbb{R}.$
- ④ $\forall a \geq 0; |x| \leq a \Leftrightarrow -a \leq x \leq a.$
- ⑤ $|x \cdot y| = |x| \cdot |y|, \forall x, y \in \mathbb{R}.$
- ⑥ $\left| \frac{x}{y} \right| = \frac{|x|}{|y|}, \forall (x, y) \in \mathbb{R} \times \mathbb{R}^*.$
- ⑦ $|x + y| \leq |x| + |y|, \forall x, y \in \mathbb{R},$ (The triangular inequality).
- ⑧ $||x| - |y|| \leq |x - y|, \forall x, y \in \mathbb{R},$ (The second triangular inequality)

Proof: (of the two last properties)

• $\forall x, y \in \mathbb{R}$

$$\begin{cases} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{cases} \quad (1)$$

Hence, by adding them up

$$-(|x| + |y|) \leq x + y \leq |x| + |y| \Leftrightarrow |x + y| \leq |x| + |y|.$$

• $\forall x, y \in \mathbb{R}$

$$|x| = |x - y + y| \Rightarrow |x| \leq |x - y| + |y| \Leftrightarrow |x| - |y| \leq |x - y|$$

and

$$|y| = |y - x + x| \Rightarrow |y| \leq |y - x| + |x| \Leftrightarrow -|x - y| \leq |x| - |y|$$

Therefore,

$$-|x - y| \leq |x| - |y| \leq |x - y| \Leftrightarrow ||x| - |y|| \leq |x - y|. \square$$

The Floor function (Integer part):

Definition: The floor function of a real number x is **the largest integer n that is less than or equal to x** . In other words, the floor of x is the unique integer n in $\mathbb{Z}$ such that $n \leq x < n + 1$. It is denoted as $\lfloor x \rfloor$ or $E(x)$. Thus, every real number x can be uniquely expressed as $x = \lfloor x \rfloor + \alpha$, where $\alpha \in [0, 1[$.

Example: $\lfloor 4.60811 \rfloor = 4$, $\lfloor -4.60811 \rfloor = -5$

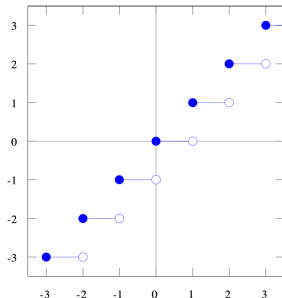


Figure: Source: <https://upload.wikimedia.org>

Properties:

- $[x] \in \mathbb{Z}$, for all $x \in \mathbb{R}$.
- $[x] \leq x \leq [x] + 1$, for all $x \in \mathbb{R}$.
- $[x + m] = [x] + m$, for all $x \in \mathbb{R}$ and $m \in \mathbb{Z}$.
- $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$, for all $x, y \in \mathbb{R}$.
- $x \leq y \Rightarrow [x] \leq [y]$, for all $x, y \in \mathbb{R}$.

Proof:

- For any $x \in \mathbb{R}$, we have

$$[x] \leq x \leq [x] + 1,$$

which implies

$$[x] + m \leq x + m \leq [x] + m + 1, \quad (2)$$

for all $m \in \mathbb{Z}$. On the other hand,

$$[x + m] \leq x + m \leq [x + m] + 1, \quad (3)$$

since $[x + m]$ is the largest integer less than $x + m$, we have

$$[x] + m \leq [x + m] \quad (4)$$

Similarly, $[x + m] + 1$ is the smallest integer greater than or equal to $x + m$, so

$$[x + m] + 1 \leq [x] + m + 1 \quad (5)$$

Combining these, we get

$$[x + m] \leq [x] + m \quad (6)$$

From $[x] + m \leq [x + m]$ and $[x + m] \leq [x] + m$, we conclude $[x + m] = [x] + m$. $\square$

Note: The floor function is increasing function but not strictly increasing.

Intervals in $\mathbb{R}$:

Definition: A subset I of $\mathbb{R}$ is an interval in $\mathbb{R}$, if whenever it contains two real numbers a and b , so that it contains all real numbers lying between them.

$$\forall a, b \in I, \forall x \in \mathbb{R}; a \leq x \leq b \Rightarrow x \in I.$$

Examples:

- $\mathbb{R}$ and the empty set $\emptyset$ are intervals.
- $\mathbb{R}^+$ is an interval.
- $\mathbb{R}^*$ and $\mathbb{N}$ are not intervals.

Remarks:

- For the notations, let $a, b \in \mathbb{R}$, we have the intervals in $\mathbb{R}$:
 - Bounded: open $]a, b[$, closed $[a, b]$, or half-open intervals $[a, b[$, $]a, b]$.
 - Unbounded: open $] - \infty, b[$, $]a, +\infty[$, or closed $[a, +\infty[$, $] - \infty, b]$.
 - If $a = b$, then $[a, a] = \{a\}$, $]a, b[= [a, b[=]a, b] = \emptyset$.

- The complement of an open interval is closed.

Note: $\mathbb{R}$ and the empty set $\emptyset$ are the only open and closed sets of $\mathbb{R}$.

Indeed, $\mathbb{R} =]-\infty, +\infty[$ is an open interval, so its complement, the empty set $\emptyset$ is closed. Moreover, the empty set $\emptyset$ can be written as an open interval $]\alpha, \alpha[$, where $\alpha \in \mathbb{R}$, so its complement $\mathbb{R}$ is closed.

Remarks:

- The intersection of two intervals is always an interval.
- The union of two intervals with a non-empty intersection is an interval.

Definition: Let $a, b \in \mathbb{R}$. We define the segment denoted as $[a, b]$ by $[a, b] = \{x \in \mathbb{R} / a \leq x \leq b\}$. If $a > b$ then $[a, b] = \emptyset$.

Definition: Let V be a subset of $\mathbb{R}$ and $x_0 \in \mathbb{R}$. We say that V is a neighborhood of x_0 if there exists an open interval $]a, b[$ in $\mathbb{R}$ containing x_0 and contained in V . We denoted this as V_{x_0} or $V(x_0)$.

Examples:

- For any $\varepsilon > 0$, the interval $V =]x_0 - \varepsilon, x_0 + \varepsilon[$ is a neighborhood of x_0 because there exists an open interval $]x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}[$ in $\mathbb{R}$ containing x_0 and contained in V .
- The interval $]a, b[$ is a neighborhood of all points $x \in]a, b[$.
- The sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are not neighborhoods of any of their points.

More About: Lower bounds, upper bounds, infimum, supremum, Maximum, Minimum

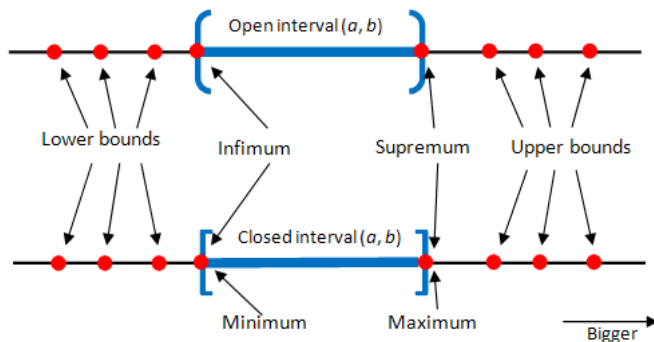


Figure: Source:<https://mathstrek.blog/2012/11/09/basic-analysis-sequence-convergence-1/>

Definition: Given a subset $E \subset \mathbb{R}$ totally ordered by the relation denoted as " $\leq$ ", and let $A \subset E$ be a non-empty subset of E .

- We say that $M \in E$ is an upper bound of A if:
 $\forall x \in A, x \leq M$.
- We say that $m \in E$ is lower bound of A if: $\forall x \in A, m \leq x$.
- A is said to be bounded above (respectively, bounded below) if it has at least one upper bound (respectively, lower bound).

Note: If A has an upper bound (respectively, lower bound), it is not necessarily unique.

Definition:

- Given a non-empty and bounded above subset A of E , and let $Maj(A) \subset E$ be set of upper bounds A , we say that $M \in E$ is the supremum of A if M is the smallest of the upper bounds of A , and we denote it as $\sup A$.
- Given a non-empty and bounded below subset A of E , and let $Min(A) \subset E$ be the set of lower bounds of A , we say that $m \in E$ is the infimum of A if m is the largest lower bounds of A , and we denote it as $\inf A$.

Remarks:

- When the supremum (respectively, infimum) exists, it is unique.
- The supremum $\sup A$ (respectively, infimum $\inf(A)$) does not necessarily belong to the set A .

Definition:

- We say that M is the greatest element of A or the maximum of A if M is an upper bound of A that belongs to A , and we denote it as $\max A$.
- We say that m is the smallest element of A that belongs to A , and we denote it as $\min A$.

Remarks:

- If the maximum $\max A$ (respectively, minimum $\min A$) exists, then $\sup A = \max A$ (respectively $\inf A = \min A$).
- If the supremum $\sup A$ (respectively, infimum $\inf A$) belongs to A , then $\max A = \sup A$ (respectively, $\min A = \inf A$).
- If the supremum $\sup A$ (respectively, infimum $\inf A$) does not belong to A , then the maximum $\max A$ (respectively, minimum $\min A$) does not exist.

Note: The supremum of a bounded above set A (respectively, the infimum of a bounded below set A) always exist but may not belong to A . On the other hand, the maximum of a bounded above set (respectively, the minimum of a bounded below set) may not exist.

Example: Let $A =] - 7, 2]$; A is a bounded subset of $\mathbb{R}$. The set of upper bounds of A is $Maj(A) = [2, +\infty[$ with $\sup A = \max A = 2$. The set of lower bounds of A is $Min(A) =] - \infty, -7]$, with $\inf A = -7$; however, $\min A$ does not exist because $-7 \notin A$.

Proposition: Let A be non-empty subset of $\mathbb{R}$. The following two statements are equivalent:

- (i) $\exists \alpha > 0, \forall x \in A : |x| \leq \alpha$.
- (ii) $\exists m, M \in \mathbb{R}, \forall x \in A : m \leq x \leq M$.

Proof:

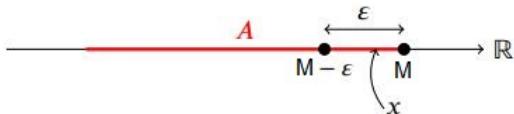
- (i) $\Rightarrow$ (ii) it suffices to take $m = -\alpha$ and $M = \alpha$.
- (ii) $\Rightarrow$ (i) it suffices to take $\alpha = \max(M, -m)$, indeed,
 $-\alpha \leq m \leq x \leq M \leq \alpha \Rightarrow -\alpha \leq x \leq \alpha \iff |x| \leq \alpha$.

Characterization of the supremum and of the infimum:

Given a non-empty and bounded set A in $\mathbb{R}$, let m and M be real numbers. We have the following characterizations:

① $M = \sup A$ if and only if:

- For all $x \in A$; $x \leq M$.
- For all $\epsilon > 0$; there exists $x \in A$ such that $M - \epsilon < x$.



② $m = \inf A$ if and only if:

- For all $x \in A$; $m \leq x$.
- For all $\epsilon > 0$; there exists $x \in A$ such that $x < m + \epsilon$.



Proof:

- ①
 - First, let's show that if $M = \sup A$, then for any $\epsilon > 0$, there exists $x \in A$ such that $M - \epsilon < x$.
We assume by contradiction that there exists $\epsilon > 0$ such that for all $x \in A$; $x \leq M - \epsilon$. Consequently, $M - \epsilon$ becomes an upper bound for A . However, since M is least upper bound of A , we have: $M \leq M - \epsilon \iff \epsilon \leq 0$, which is a contradiction.
 - Now let's show that if M is an upper bound of A and satisfies $\forall \epsilon > 0; \exists x_0 \in A, M - \epsilon < x_0$, then M is the least upper bound of A . Let M_0 be another upper bound of A , hence $x_0 \leq M_0$.
Therefore: $\forall \epsilon > 0; M - \epsilon < x_0 \leq M_0 \Rightarrow \forall \epsilon > 0; M - M_0 < \epsilon$, which implies $M - M_0 < \epsilon$, which implies $M - M_0 \leq 0 \iff M \leq M_0$.
- ② The characterization of the lower bound can be shown in a similar manner. $\square$

Properties:

- ① Given two non-empty, bounded sets A and B in $\mathbb{R}$, such that $A \subset B$, then:

$$\inf B \leq \inf A \leq \sup A \leq \sup B. \quad (7)$$

In fact, we have

- $\inf A \leq x \leq \sup A$ for all $x \in A$, which implies $\inf A \leq \sup A$.
- For every x in A , we also have $x \in B$, which leads to $\inf B \leq x$ for all x in A .

This means $\inf B$ is a lower bound for A , but $\inf A$ is the greatest lower bound for A , so $\inf B \leq \inf A$. Furthermore,

- for all x in A (and hence x in B), we have $x \leq \sup B$,

which means $\sup B$ is an upper bound for A . But $\sup A$ is the smallest upper bound for A , so $\sup A \leq \sup B$.

② Given C and D , two non-empty, bounded sets in $\mathbb{R}$, then:

- $\sup(C \cup D) = \max(\sup C, \sup D)$
 $\inf(C \cup D) = \min(\inf C, \inf D)$
- $\sup(C \cap D) \leq \min(\sup C, \sup D)$
 $\inf(C \cap D) \geq \max(\inf C, \inf D)$
- $\sup(C + D) = \sup C + \sup D$
 $\inf(C + D) = \inf C + \inf D$
where $C + D = \{x + y / x \in C, y \in D\}$

The density of $\mathbb{Q}$ in $\mathbb{R}$:

Theorem: Given two distinct real numbers a and b such that $a < b$, then the interval $]a, b[$ contains at least one rational number $q \in \mathbb{Q}$. We say that $\mathbb{Q}$ is dense in $\mathbb{R}$ and denote it as $\bar{\mathbb{Q}} = \mathbb{R}$.

Thanks