Series 1: Mathematical Language and Proof Writing

Exercises

1. **Types of Mathematical Statements**
   Identify whether the following statements are best described as an **Axiom**, **Theorem**, **Lemma**, **Corollary**, **Proposition**, or **Conjecture**.

   (a) The sum of the angles in any triangle is 180 degrees.

   (b) Every even integer greater than 2 can be expressed as the sum of two prime numbers. (Goldbach's Conjecture)

   (c) If two parallel lines are cut by a transversal, then the alternate interior angles are equal.

   (d) A preparatory result used to prove that every natural number greater than 1 has a prime divisor.

   (e) The statement "0 is a natural number" in the Peano axioms.

2. **Writing Definitions**
   Write a precise mathematical definition for the following concepts:

   (a) An **even** integer.

   (b) A **rational** number.

   (c) A **prime** number.

3. **Direct Proof**
   Prove that the sum of two even integers is even.

4. **Proof by Contrapositive**
   Prove the following statement by contrapositive:
   "If n^2 is even, then n is even." (Assume n is an integer.)

5. **Proof by Contradiction**
   Prove that $\sqrt{3}$ is irrational.

6. **Proof by Induction**
   Prove by induction that for every positive integer n:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

7. **Using "Without Loss of Generality" (WLOG)**
   Prove that if x and y are two real numbers with $x \neq y$, then $\frac{x+y}{2}$ is strictly between x and y. Use WLOG to simplify the proof.

8. **Constructive vs. Non-Constructive Proof**

   (a) Give a **constructive** proof that there exist two distinct irrational numbers a and b such that a^b is rational.

   (b) Explain why the proof of Proposition 1.1 in the text is **non-constructive**.

Solutions

1. **Types of Mathematical Statements**
   (a) Theorem
   (b) Conjecture
   (c) Theorem (or Axiom, depending on the geometric system)
   (d) Lemma
   (e) Axiom

2. **Writing Definitions**
   (a) An integer n is **even** if there exists an integer k such that $n = 2k$.
   (b) A number r is **rational** if there exist integers a and b with $b \neq 0$ such that $r = \frac{a}{b}$.
   (c) A natural number $p > 1$ is **prime** if its only positive divisors are 1 and itself.

3. **Direct Proof**
   Let m and n be even integers. Then there exist integers a and b such that $m = 2a$ and $n = 2b$.
   Their sum is:
   $$m + n = 2a + 2b = 2(a + b)$$
   Since $a + b$ is an integer, $m + n$ is even.

4. **Proof by Contrapositive**
   The contrapositive of "If n^2 is even, then n is even" is:
   "If n is odd, then n^2 is odd."
   Assume n is odd. Then $n = 2k + 1$ for some integer k.
   Then:
   $$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$
   This shows that n^2 is odd. Hence, the contrapositive is true, so the original statement is true.

5. **Proof by Contradiction**
Assume, for contradiction, that $\sqrt{3}$ is rational. Then there exist coprime integers a and b (with $b \neq 0$) such that:

$$\sqrt{3} = \frac{a}{b}$$

Squaring both sides:

$$3 = \frac{a^2}{b^2} \quad \Rightarrow \quad a^2 = 3b^2$$

So a^2 is divisible by 3, which implies a is divisible by 3. Let $a = 3k$. Then:

$$(3k)^2 = 3b^2 \quad \Rightarrow \quad 9k^2 = 3b^2 \quad \Rightarrow \quad b^2 = 3k^2$$

So b^2 is divisible by 3, which implies b is divisible by 3. But this contradicts the assumption that a and b are coprime. Hence, $\sqrt{3}$ is irrational.

6. **Proof by Induction**
Let $P(n)$ be the statement:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Base Case: For $n = 1$, LHS $= 1$, RHS $= \frac{1 \cdot 2}{2} = 1$. So $P(1)$ is true.
Inductive Step: Assume $P(k)$ is true for some $k \geq 1$. That is:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

We must show $P(k+1)$ is true:

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Starting from the LHS and using the inductive hypothesis:

$$\frac{k(k+1)}{2} + (k+1) = (k+1)\left(\frac{k}{2} + 1\right) = (k+1) \cdot \frac{k+2}{2}$$

This is the RHS of $P(k+1)$. So $P(k+1)$ is true.
By induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

7. **Using "Without Loss of Generality" (WLOG)**
   Assume $x < y$. (We can do this without loss of generality because if $y < x$, we could simply relabel the variables.)
   We want to show:
   $$x < \frac{x+y}{2} < y$$
   First, $x < \frac{x+y}{2} \iff 2x < x + y \iff x < y$, which is true.
   Second, $\frac{x+y}{2} < y \iff x + y < 2y \iff x < y$, which is also true.
   Hence, $\frac{x+y}{2}$ lies strictly between x and y.

8. **Constructive vs. Non-Constructive Proof**

   (a) **Constructive proof:**
   Let $a = \sqrt{2}$ and $b = \log_2 9$. Both are irrational. Then:
   $$a^b = \sqrt{2}^{\log_2 9} = 2^{\frac{1}{2}\log_2 9} = 2^{\log_2 3} = 3$$
   which is rational.

   (b) The proof of Proposition 1.1 is **non-constructive** because it shows that such numbers x and y must exist (by considering $\sqrt{2}^{\sqrt{2}}$), but it does not explicitly give a pair (x, y) that works, since it depends on whether $\sqrt{2}^{\sqrt{2}}$ is rational or not — a fact not determined by the proof itself.