Chapter 1: Mathematical Foundations of Complex Numbers (CN)

1. Definition:

The equation $x^2 + 1 = 0$ has no solution in the set of real numbers $\mathbb{R}$. Solving it requires the square roots of a negative number, which led to the invention of complex numbers. In $\mathbb{C}$, the set of complex numbers, it has two solutions: i and $-i$, with $i^2 = -1$.
The notation i was introduced by Euler, the great Swiss mathematician. In this document, however, we will use j instead of i. In electrical engineering, i is commonly used to denote electric current intensity [1].

2. Algebraic Form of a Complex Number

Let a complex number $z = a + jb$, with a and b being real numbers. The expression $a + jb$ is called the algebraic form of z, where:

$a = Re(z)$ Represents the real part of z,

$b = Im(z)$ Represents the imaginary part of z.

These two values (a and b) can be taken as coordinates in the Cartesian plane (see Figure 1) [2].

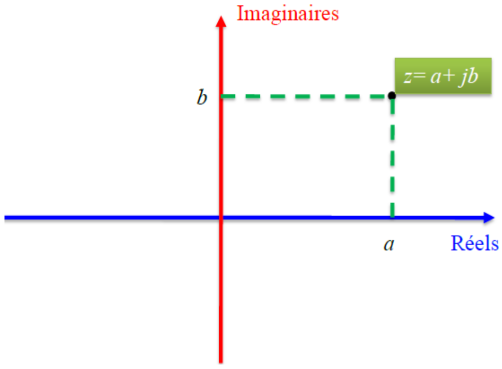

Figure 1: Cartesian Representation of a Complex Number (z).

3. Geometric Form of a Complex Number

In a plane equipped with an orthonormal coordinate system O, u, v, each complex number $z = a + jb$ is associated with the point $P(a, b)$. Conversely, to every point in the plane, one can associate a complex number.

$P(a, b)$ is the image of $z = a + jb$, and z is the affix of the vector $\overrightarrow{OP} = \overrightarrow{au} + \overrightarrow{bv}$.

4. Trigonometric Form of a Complex Number

Let P be the image of $z = a + jb$ in the plane with respect to the coordinate system O, u, v.

<u>The modulus</u> of z is the positive real number $r = |z| = \sqrt{a^2 + b^2}$.

<u>The argument</u> of z ($z \neq 0$) is the angle θ, defined modulo $2k\pi$ (k in $\mathbb{Z}$) with :

$$\begin{cases} \sin(\theta) = \dfrac{b}{r} \\ \cos(\theta) = \dfrac{a}{r} \\ \theta = tang^{-1}\left(\dfrac{b}{a}\right) \end{cases}$$

Geometrically θ is up to $2k\pi$, the measure of the angle $(\vec{u}, \overrightarrow{OP})$. The trigonometric form of z is then: $z = \rho(\cos(\theta) + isin(\theta))$.

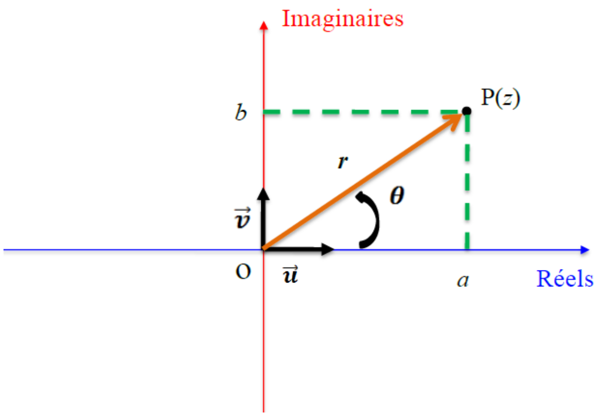

Figure 2 : Trigonometric Representation of a Complex Number (z) [2].

5. Polar Representation of a Complex Number (z)

Let z be a complex number ($z \neq 0$) with r and θ denoting the modulus and the argument, respectively. The polar representation is given in the following form:

$$z = [r, \theta]$$

6. Exponential Representation of a Complex Number

Euler's formula connects the complex exponential with cosine and sine in the complex plane:

$$\forall\, \theta \in \mathbb{R}, \qquad e^{j\theta} = cos(\theta) + jsin(\theta)$$

Theorem

Let θ be a real number. Euler's formula is expressed as:

$$\begin{cases} \cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j} \end{cases}$$

Proof

We have:

$e^{j\theta} = \cos\theta + j\sin\theta$ and $e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta$.

Thus, we have: $\begin{cases} e^{j\theta} = \cos\theta + j\sin\theta \\ e^{-j\theta} = \cos\theta - j\sin\theta \end{cases}$

By adding and subtracting the two equalities term by term, we obtain Euler's formulas.
A complex number can then be written in exponential form, as shown in Figure 3 [2]:

$$z = re^{j\theta}$$

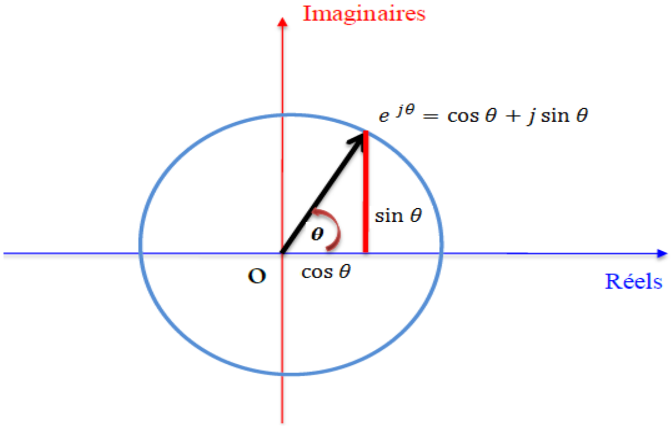

Figure 3 : Exponential Form of a Complex Number (CN).

Moivre's Formula

The $n - th$ power of a complex number z is given by Moivre's formula:

$$(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta)$$

Proof

For a complex number with modulus $r = 1$, we can write z in the form:

$$z = e^{j\theta}$$

7. Operations on Complex Numbers

Let a complex number $z = a + jb$. The conjugate of z, denoted $\bar{z}$, is the same number but with the opposite imaginary part: $\bar{z} = a - jb$

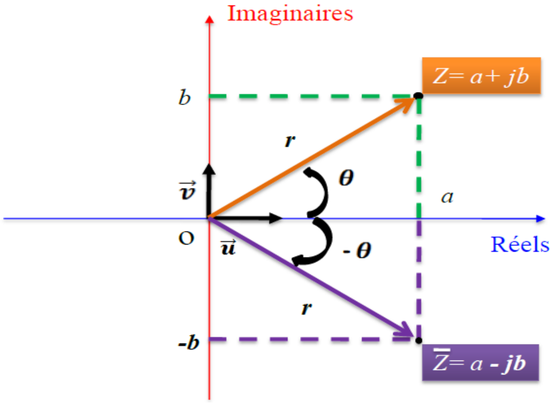

Figure 4: Complex Conjugate.

Addition

The addition of two complex numbers in algebraic form $z = a + jb$ and $z' = a' + jb'$ consists in adding the real part to the real part, and the imaginary parts to each other.

$$z + z' = (a + a') + j(b + b')$$

By applying the same reasoning, the subtraction of complex numbers is given by:

$$z - z' = (a - a') + j(b - b')$$

Product

Let two complex numbers z and z', Their product is given by:

$$z * z' = (a + jb) * (a' + jb')$$
$$= (aa' - bb') + j(ab' + a'b)$$

For the product, it is often more convenient to use the polar form $\begin{cases} z = [r_1,\ \theta_1] \\ z' = [r_2,\ \theta_2] \end{cases}$

The product of these two numbers in polar form is then:

$$z * z' = [(r_1 * r_2), (\theta_1 + \theta_2)]$$

Division

The calculation of the division of two complex numbers z and z' in algebraic form requires the conjugate of the denominator.

$$\frac{a + jb}{a' + jb'} = \frac{(a + jb) * (a' - jb')}{(a' + jb') * (a' - jb')} = \frac{(aa' + bb') + j(a'b - ab')}{a'^2 + b'^2}$$

For the division, it is often more convenient to use the polar form:

$$\frac{z}{z'} = [\left(\frac{r_1}{r_2}\right), (\theta_1 - \theta_2)]$$

Application of complex numbers to electricity

Let an alternating sinusoidal current $i(t)$ whose expression as a function of time is given by:

$$i(t) = I\sqrt{2}\sin(\omega t + \varphi)$$

To this sinusoidal quantity, we associate a Fresnel vector. We can then associate with $i(t)$ a complex representation:

$$\underline{I} = [I, \varphi] = I(\cos\varphi + j\sin\varphi)$$

Similarly, for the applied voltage, we can write:

$$\underline{U} = [U, \varphi] = U(\cos\varphi + j\sin\varphi)$$

It should be noted that the complex representation is very effective for calculating the parameters of electrical circuits [3].

References

[1] James Ward Brown, Ruel V. Churchill. "Complex Variables and Applications", Eighth Edition, ISBN 978–0–07–305194–9.2008

[2] Mohammed Amine ZAFRANE. "Electrotechnique Fondamentale 1 (Cours et applications) ", USTO-MB, pp 1-93, 2020

[3] http://pagesperso-orange.fr/fabrice.sincere/