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NOTATION ET TERMINOLOGIE

$\mathbb{R}^n$	:	L'ensemble des vecteurs avec n composants.
$\mathbb{R}_+^n$	:	L'orthant positif de l'espace $\mathbb{R}^n$.
$\mathbb{R}^{n \times m}$	:	L'espace vectoriel des matrices réelles de taille $(n \times m)$
$\mathbb{R}_{++}^n$	:	l'orthant strictement positif de l'espace $\mathbb{R}^n$.
$s.c$	:	Sous les contraintes.
x^*	:	La solution optimale du problème.
$\emptyset$	:	L'ensemble vide.
x^t	:	Le transposé du vecteur x de $\mathbb{R}^n$.
(PM)	:	Programmation mathématique.
(PL)	:	Programmation linéaire.
(DL)	:	Le problème dual de Programmation linéaire.
K.K.T	:	Karush-Kuhn-Tucker.
$\Delta x, \Delta y, \Delta s$	:	les directions de Newton.
e	:	le vecteur de $\mathbb{R}$, dont toutes les composantes sont égales à 1.
$\psi(t)$	:	Fonction noyau.
$\phi(v)$	$= \sum_{i=1}^n \psi(v_i)$	:La fonction barrière logarithmique de type primal-dual.

INTRODUCTION PROBABILITY AND STATISTICS

CHAPITRE 1

COMBINATORIAL ANALYSIS

1.1

EXEMPLE

How many ways to order 52 cards ?

Answer : $52 \cdot 51 \cdot 50 \dots 1 = 52!$

n hats, n people, how many ways to assign each person a hat ?

Answer : $n!$

n hats, $k < n$ people, how many ways to assign each person a hat ?

$n(n-1)(n-2)\dots(n-k+1) = n!/(n-k)!$

1.1.1 Arrangements

Définition 1.1.1 [3] Let E be a set with n elements, an arrangement of p of these objects is an ordered sequence of p objects taken from these n objects.

There are two types of arrangements : with and without repetition.

Arrangement without repetition

We call an arrangement without repetition of p objects chosen from n objects any ordered layout (disposition) of p objects taken from the n objects without repetitions.

The number of arrangements without repetition, noted A_p^n , is as follows :

$$A_p^n = \frac{n!}{(n-p)!} = n \times (n-1) \times (n-2) \cdots \times (n-p+1),$$

where $1 \leq p \leq n$.

In an arrangement without repetition, the p objects in the list are all distinct. This corresponds to a draw without replacement and with order.

Example How many three-letter words containing no more than one letter can be formed using the letters of the alphabet ?

$$A_{26}^3 = \frac{26!}{(26-3)!} = 26 \times 25 \times 24 = 15600 \text{ mots.}$$

Arrangement with repetition

We call an arrangement with repetition of p objects chosen from n objects any ordered layout (disposition) of p objects taken from the n objects with repetitions.

The number of arrangements with repetition, noted n^p , is as follows :

$$n^p = n \times n \times n \cdots \times n,$$

where $1 \leq p \leq n$.

In a non-repetition arrangement, the p objects in the list are not necessarily all distinct. This corresponds to a draw with replacement and with order.

Example How many two-letter words can be made with the letters of the alphabet?

$$26^2 = 26 \times 26.$$

1.1.2 Permutations

Définition 1.1.2 [3] *Let E a set of n objects. We call permutation of n distinct objects any ordered sequence of n objects or any arrangement n to n of these objects.*

Permutation without repetition

This is the special case of the arrangement without repetition of p objects among n objects, when $p = n$.

The number of permutations of n objects is : $n!$

Example The number of ways to seat eight diners (guests) around a table is : $8! = 40320$.

Permutation with repetition

In the case where there are k identical objects among the n objects, then

$$\frac{n!}{k!}$$

Example The number of possible words (with or without meaning) that can be formed by permuting the 8 letters of the word "Quantity" is $\frac{8!}{2!} = 20160$ words, we have 2 t in "Quantity".

Considering the word "Swimming", the number of possible words is $\frac{8!}{2!2!} = 10080$ words,

because we have the i 2 times and the m 2 times.

1.1.3 Combinations

Combination without repetitions (without discounts)

Définition 1.1.3 [3] *Given a set E of n objects. We call combinations of p objects any set of p objects taken from the n objects without replacement (without discount).*

The number of combinations of p objects among n and without replacement, is :

$$C_n^p = \frac{n!}{p!(n-p)!}$$

where $1 \leq p \leq n$.

Example 1 The random drawing of 5 cards from a deck of 32 cards (poker hand) is a combination with $p = 5$ and $n = 32$. The number of possible drawings is : $C_{32}^5 = \frac{32!}{5!(32-5)!} = 409696$ possibilities.

Example 2 Forming a delegation of 2 students from a group of 20 is a combination with $p = 2$ and $n = 20$. The number of possible delegations is $C_{20}^2 = \frac{20!}{2!(20-2)!} = 190$ possibilities.

Combination with repetitions (with discounts)

The number of combinations of p objects among n and with replacement (with discount), is :

$$C_{n+p-1}^p = \frac{(n+p-1)!}{p!(n-1)!}$$

where $1 \leq p \leq n$.

Example1 Let's make up 3-letter words from a 5-letter alphabet with discount.

The number of words is $C_{5+3-1}^3 = C_7^3 = 35$.

There are 3 possible cases :

- C_5^3 number of words of 3 different letters and without order ;

$-2C_5^2$ number of words with 2 different letters and one redundant letter ;

$-C_5^1$ number of words with 3 identical letters ;

in total, we have $C_5^3 + 2C_5^2 + C_5^1 = C_7^3 = 35$ words.

Let E be a set with n elements. p -lists : We call a p -list of E any ordered sequence (the order is important) of p elements taken from n elements of E . n_p : number of p -lists of E . Arrangements : On appelle arrangement de p éléments, toute suite ordonnée (l'ordre est important) de p éléments distincts pris parmi n éléments de E . : number of arrangements of p elements among n elements of E . Permutations : If $p = n$, the arrangements of n elements among n elements will be called permutations of n elements. (A permutation is a map from $1, 2, \dots, n$ to $1, 2, \dots, n$. There are $n!$ permutations of n elements.) $P_n = A_p n = n!$: number of permutations of n elements. Combinations : A combination is a p -element subset of n elements of E . Here, the order is not important and repetition of elements is prohibited. : number of combinations of p elements among n elements of E .

CHAPITRE 2

PROBABILITY SPACES

2.1 Sample spaces and events

Définition 2.1.1 (*Random Experiment :*) *An experiment whose outcome is uncertain before it is performed is called a random experiment.*

Définition 2.1.2 (*Sample Space*) *The set of all possible outcomes of the given experiment is called the sample space and is denoted by Ω , and an element of Ω by ω .*

Examples

- Coin toss : $\Omega = \{Heads, Tails\} = \{H, T\}$
- Roll of a die : $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Tossing of two coins : $\Omega = \{(H, H), (T, H), (H, T), (T, T)\}$
- Coin is tossed until heads appear. What is Ω ? - Life expectancy of a random person.
 $\Omega = [1, 120]$ years

Définition 2.1.3 (*Events*) *An event, A , is a subset of the sample space.*

This means that event A is simply a collection of outcomes. Events are typically denoted by upper case letters, usually from the beginning of the alphabet.

An event is said to have occurred if the outcome of the experiment belongs to it.

Examples

- sure event = sample space Ω (an event for sure to occur)
- impossible event = empty set $\emptyset$ (an event impossible to occur)
- Coin toss : $\Omega = \{Heads, Tails\}$

$E = \{Heads\}$ is the event that a head appears on the flip of a coin.

- Roll of a die : $\Omega = \{1, 2, 3, 4, 5, 6\}$. Some possible events are

$E_1 = \{1\}$ (single outcome), $E_2 = \{an\ even\ number\ shows\ up\} = \{2, 4, 6\}$ (3 outcomes),

$E_3 = \{the\ outcome\ is\ \geq\ 3\} = \{3, 4, 5, 6\}$, $E_4 = \{the\ outcome\ is\ \leq\ 0\} = \emptyset$, (no outcome).

- Life expectancy. $\Omega = [1, 120]$.

$E = [50, 120]$ is the event that a random person lives beyond 50 years.

2.1.1 Language of Events

Typical Notation	Language of Sets	Language of Events
Ω	Whole space	Certain event
$\emptyset$	Empty set	Impossible event
A	Subset of Ω	Event that some outcome in A occurs
A^c	Complement of A	Event that no outcome in A occurs
$A \cup B$	Union	Event that an outcome in A or B or both occurs
$A \cap B$	Intersection	Event that an outcome in both A and B occurs
$A \cap B = \emptyset$	Disjoint sets	Mutually exclusive events

Définition 2.1.4 Events A and B are disjoint if their intersection is empty,

$$A \cap B = \emptyset.$$

Events $A_1, A_2, \dots, A_n$ are mutually exclusive or pairwise disjoint if any two of these events are disjoint, i.e.,

$$A_i \cap A_j = \emptyset \text{ for any } i \neq j.$$

Définition 2.1.5 (sigma-algebra (σ -algebra) (σ -field)) A collection $\mathcal{F}$ of subsets of Ω is called a sigma-algebra if it satisfies

(a) it includes the sample space : $\Omega \in \mathcal{F}$

(b) $\mathcal{F}$ is stable by countable union, i.e. if $A_1, A_2, \dots \in \mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$

(c) $\mathcal{F}$ is stable by complement, if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

Examples - $\mathcal{F} = \{\Phi, \Omega\}$ is the smallest sigma-algebra (Degenerate sigma-algebra).

- If $A \subset \Omega$, $\mathcal{F} = \{\Phi, A, A^c, \Omega\}$ is a σ -field

- $\mathcal{P}(\Omega)$ is the richest sigma-algebra. This sigma-algebra is called a power set (is the set whose elements are all the subsets of Ω).

- the Borel sigma-algebra, if $\Omega = \mathbb{R}$, and is denoted by $\mathcal{B}_{\mathbb{R}}$

Définition 2.1.6 (*Indicator function*) Indicator function of an event denoted by $I_A(\omega)$ and defined as

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{si } \omega \notin A \end{cases}$$

Définition 2.1.7 (*Partition of sample space*) A partition of a set A is a set $\{A_1, A_2, \dots, A_n\}$ with the following properties :

a. $A_i \subseteq A$, $i = 1, 2, \dots, n$, which means that A is a set of subsets.

b. $A_i \cap A_j = \emptyset$, for every $i \neq j$, $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$, which means that the subsets are mutually (or pairwise) disjoint ; that is, no two subsets have any element in common.

c. $\cup_{i=1}^n A_i = A$, which means that the subsets are collectively exhaustive. That is, the subsets together include all possible values of the set A .

Définition 2.1.8 (*Probability and probability space*) Given a σ -algebra $\mathcal{F}$ on a set Ω . A probability $\mathbb{P}$ is a function $\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$ satisfying (s.t.)

(a) (Unit measure) The sample space has unit probability, $P(\Omega) = 1$

(b) (Sigma-additivity) if $A_1, A_2, \dots \in \mathcal{F}$ is a collection of disjoint members in $\mathcal{F}$, then

$$\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

(P is said to be countably additive)

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

Remarque For any event A ,

$$\mathbb{P}(A) = \frac{\text{The number of outcomes in } A}{\text{The number of outcomes in } \Omega}$$

Examples 1

Which of the following are Probability functions ?

(i) $\Omega = \{1, 2, 3, \dots\}$, $\mathcal{F}$ is σ - field on Ω . A function $\mathbb{P}$ defined on space $(\Omega, \mathcal{F})$ as

$$\mathbb{P}(i) = \frac{1}{2^i}$$

for $i \in \Omega$

Solution

a)

$$\mathbb{P}(\Omega) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

b)

$$\mathbb{P}(A) \geq 0 \text{ for all } A \in \mathcal{F}$$

c) Let us define mutually exclusive events, $A_i = i$ we can verify countable additivity.

$$\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

By a), b) and c) $\mathbb{P}$ is Probability function.

Examples 1

- Coin toss : $\Omega = \{H, T\}$, $\mathcal{F} = \{\emptyset, H, T, \Omega\}$

$$\mathbb{P}(\Omega) = 1, \mathbb{P}(\emptyset) = 0, \mathbb{P}(H) = \frac{1}{2}, \mathbb{P}(T) = \frac{1}{2}$$

- Roll of a die : $\Omega = \{1, 2, 3, 4, 5, 6\}$.

The probability of obtaining the 1 is $1/6$. The probability of obtaining the 2 is $1/6$. In fact, the probability of obtaining any particular integer from $1, 2, \dots, 6$, is $1/6$.

I defined the events $A = \{5, 6\}$, and $B = \{1, 3, 5\}$. We can now see that $\mathbb{P}(A) = 2/6$ and $\mathbb{P}(B)$

$= 3/6$.

Examples 2

Two fair dice are tossed. Find the probability of each of the following events :

- a. The sum of the outcomes of the two dice is equal to 7.
- b. The sum of the outcomes of the two dice is equal to 7 or 11.
- c. The outcome of the second die is greater than the outcome of the first die.
- d. Both dice come up with even numbers.

Solution

We first define the sample space of the experiment. If we let the pair (x, y) denote the outcome ?first die comes up x and second die comes up y , ? where $x, y \in \{1, 2, 3, 4, 5, 6\}$, then $\Omega = \{(1, 1), (1, 2), (1, 3), \dots, (6, 6)\}$. The total number of sample points is 36.

(a) Let A_1 denote the event that the sum of the outcomes of the two dice is equal to seven. Then $A_1 = (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$. Since the number of sample points in the event is 6, we have that $P(A_1) = 6/36 = 1/6$.

(b) Let B denote the event that the sum of the outcomes of the two dice is either seven or eleven, and let A_2 denote the event that the sum of the outcomes of the two dice is eleven. Then, $A_2 = (5, 6), (6, 5)$ with 2 sample points. Thus, $P(A_2) = 2/36 = 1/18$. Since B is the union of A_1 and A_2 , which are mutually exclusive events, we obtain $P(B) = P(A_1 \cup A_2) = P(A_1) + P(A_2) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}$

(c) Let C denote the event that the outcome of the second die is greater than the outcome of the first die. Then $C = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6)\}$ with 15 sample points. Thus, $P(C) = 15/36 = 5/12$.

(d) Let D denote the event that both dice come up with even numbers. Then $D = (2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)$ with 9 sample points. Thus, $P(D) = 9/36 = 1/4$.

Proposition 2.1.1 1- $\mathbb{P}(\emptyset) = 0$.

2- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

3- if $A_1 \subset A_2$ then $\mathbb{P}(A_1) \leq \mathbb{P}(A_2)$.

4- $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$.

5- $\mathbb{P}(\cup_{i \geq 1} A_i) \leq \sum_{i \geq 1} \mathbb{P}(A_i)$.

Preuve. 1- By the same definition, $\mathbb{P}(\Omega) = \mathbb{P}(\Omega \cup \emptyset) = \mathbb{P}(\Omega) + \mathbb{P}(\emptyset)$, because Ω and $\emptyset$ are mutually exclusive. Therefore, $\mathbb{P}(\emptyset) = 0$.

2- Recall that events A and A^c are exhaustive, hence $A \cup A^c = \Omega$. Also, they are disjoint, hence

$$\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(A \cup A^c) = \mathbb{P}(\Omega) = 1.$$

Solving this for $\mathbb{P}(A^c)$, we obtain a rule that perfectly agrees with the common sense,

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

3- If $A_1 \subset A_2$, then A_2 can be written as the union of the disjoint subsets A_1 and $A_2 - A_1$. Therefore,

$$\mathbb{P}(A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2 - A_1) \geq \mathbb{P}(A_1).$$

4- $A_1 \cup A_2$ can be written as the disjoint union of $A_1 - A_2$, $A_2 - A_1$ and $A_1 \cap A_2$. Therefore,

$$\begin{aligned} \mathbb{P}(A_1 \cup A_2) &= \mathbb{P}(A_1 - A_2) + \mathbb{P}(A_2 - A_1) + \mathbb{P}(A_1 \cap A_2) \\ &= [\mathbb{P}(A_1) - \mathbb{P}(A_1 \cap A_2)] + [\mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)] + \mathbb{P}(A_1 \cap A_2) \\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2). \end{aligned}$$

■ Examples

Three tulip bulbs are planted in a window box. Find the probability that at least one will flower if the probability that all will fail to flower is $\frac{1}{8}$.

Sometimes calculations are made easier by using complementary events.

Solution

Définition 2.1.9 $f(\cdot)$ is a function defined on $\mathbb{R}$ is called as Borel function if inverse image is a Borel set.

2.2 Conditional Probability and independence

2.2.1 Conditional probability

Définition 2.2.1 *If E and F are two events associated with the same sample space of a random experiment, then the conditional probability of the event E under the condition that the event F has occurred (the conditional probability of E given F), written as $\mathbb{P}(E|F)$, is given by*

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$$

Examples

A bag contains eight red balls, four green balls, and eight yellow balls. A ball is drawn at random from the bag, and it is not a red ball. What is the probability that it is a green ball.

Solution

Let G denote the event that the selected ball is a green ball, and let $\bar{R}$ denote the event that it is not a red ball. Then, $P(G) = 4/20 = 1/5$, since there are 4 green balls out of a total of 20 balls, and $P(\bar{R}) = 12/20 = 3/5$, since there are 12 balls out of 20 that are not red. Now,

$$P(G|\bar{R}) = \frac{P(G \cap \bar{R})}{P(\bar{R})}$$

But if the ball is green and not red, it must be green. Thus, we obtain that $G \cap \bar{R} = G$ and

$$P(G|\bar{R}) = \frac{P(G \cap \bar{R})}{P(\bar{R})} = \frac{P(G)}{P(\bar{R})} = \frac{1/5}{3/5} = \frac{1}{3}$$

Properties

Let E and F be events associated with the sample space Ω of an experiment. Then :

- (1) $\mathbb{P}(\Omega|F) = \mathbb{P}(F|F) = 1$
- (2) $\mathbb{P}(A \cup B|F) = \mathbb{P}(A|F) + \mathbb{P}(B|F) - \mathbb{P}(A \cap B|F)$.
- (3) $\mathbb{P}(E^c|F) = 1 - \mathbb{P}(E|F)$
- (4) $\mathbb{P}(E \cap F) = \mathbb{P}(E|F)\mathbb{P}(F) = \mathbb{P}(F|E)\mathbb{P}(E)$.

(5) $\mathbb{P}(E|F) \neq \mathbb{P}(F|E)$

Examples

Show that

$$\mathbb{P}(A \cup B|F) = \mathbb{P}(A|F) + \mathbb{P}(B|F) - \mathbb{P}(A \cap B|F)$$

Solution

$$\mathbb{P}(A \cup B|F) = \frac{\mathbb{P}((A \cup B) \cap F)}{\mathbb{P}(F)}$$

By definition of conditional prob.

$$= \frac{\mathbb{P}((A \cap F) \cup (B \cap F))}{\mathbb{P}(F)}$$

By Distributive law

$$= \frac{\mathbb{P}(A \cap F) + \mathbb{P}(B \cap F) - \mathbb{P}(A \cap B \cap F)}{\mathbb{P}(F)}$$

By Addition theorem on probability.

$$= \mathbb{P}(A|F) + \mathbb{P}(B|F) - \mathbb{P}(A \cap B|F)$$

By definition of conditional prob.

Examples

Probability that it rains today is 0.4; probability that it will rain tomorrow is 0.5, probability that it will rain tomorrow and rains today is 0.3.

Given that it has rained today, what is the probability that it will rain tomorrow ?

Solution

Denote the events, say

A : "it rains today", $\mathbb{P}(A) = 0.4$

B : "it will rain tomorrow", $\mathbb{P}(B) = 0.5$, $\mathbb{P}(A \cap B) = 0.3$.

Required probability is

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = 0.75$$

Proposition 2.2.1 *If $\{A_1, A_2, \dots, A_n\}$ form a partition of Ω . Let A be any event. Then*

$$\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(A \cap A_i) = \sum_{i=1}^n \mathbb{P}(A_i)P(A|A_i)$$

Preuve. regarde chapter 1 (Basic Probability Concepts) ■ As a special case, B and $\bar{B}$ is a partition of Ω , so :

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \bar{B}) \\ &= \mathbb{P}(B)P(A|B) + \mathbb{P}(\bar{B})P(A|\bar{B}) \text{ for any } A, B.\end{aligned}$$

2.2.2 Independent Events

Now we can give an intuitively very clear definition of independence.

Définition 2.2.2 Let E and F be two events associated with a sample space Ω . If the probability of occurrence of one of them is not affected by the occurrence of the other, then we say that the two events are independent. Thus, two events E and F will be independent, if

(a) $\mathbb{P}(F|E) = \mathbb{P}(F)$.

(b) $\mathbb{P}(E|F) = \mathbb{P}(E)$.

Using the multiplication theorem on probability, we have

$$\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F).$$

Example

There is a 0,01 probability for a hard drive to crash. Therefore, it has two backups, each having a 0,02 probability to crash, and all three components are independent of each other. The stored information is lost only in an unfortunate situation when all three devices crash. What is the probability that the information is saved ?

Solution

Organize the data. Denote the events, say,

H : "hard drive crashes", B_1 = "first backup crashe", B_2 = "second backup crashes".

It is given that H , B_1 , and B_2 are independent,

$\mathbb{P}(H) = 0.01$, and $\mathbb{P}B_1 = \mathbb{P}B_2 = 0.02$.

Applying rules for the complement and for the intersection of independent events,

$$\begin{aligned}\mathbb{P}(\text{saved}) &= 1 - \mathbb{P}(\text{lost}) = 1 - \mathbb{P}(H \cap B_1 \cap B_2) \\ &= 1 - PHPB_1PB_2 \\ &= 1 - (0.01)(0.02)(0.02) = 0.999996.\end{aligned}$$

Théorème 2.2.1 *If events A and B are independent so are (i) A and $\bar{B}$ (ii) B and $\bar{A}$ (iii) $\bar{A}$ and $\bar{B}$.*

Preuve. (i) Consider

$$\mathbb{P}(A \cap \bar{B}) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$$

Since A and B are independent

$$\mathbb{P}(A \cap \bar{B}) = \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A)\mathbb{P}(\bar{B})$$

So, A and $\bar{B}$ are independent. Similarly we can prove (ii)

(iii) Consider

$$\mathbb{P}(\bar{A} \cap \bar{B}) = \mathbb{P}(\bar{A \cup B}) = 1 - \mathbb{P}(A \cup B)$$

$$\begin{aligned}&= 1 - [\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)] \\ &= [1 - \mathbb{P}(A)][1 - \mathbb{P}(B)] \\ &= \mathbb{P}(\bar{A})\mathbb{P}(\bar{B})\end{aligned}$$

So, $\bar{A}$ and $\bar{B}$ are independent ■

Remarque 2.2.2 *If A,B,C are three events*

-They are pairwise independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

$$\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$$

$$\mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$$

-They are completely independent (are said to be mutually independent) if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

$$\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$$

$$\mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$$

$$\text{and } \mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$$

Théorème 2.2.3 (Theorem of Total Probability) *Let $\{A_1, A_2, \dots, A_n\}$ be a partition of the sample space Ω . Let A be any event associated with Ω , then*

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)\mathbb{P}(A|A_i)$$

Théorème 2.2.4 (Bayes's Theorem) *If $\{A_1, A_2, \dots, A_n\}$ are mutually exclusive and exhaustive events associated with a sample space, and A is any event of non zero probability, then*

$$\mathbb{P}(A_i|A) = \frac{\mathbb{P}(A_i)\mathbb{P}(A|A_i)}{\sum_{i=1}^{\infty} \mathbb{P}(A_i)\mathbb{P}(A|A_i)}$$

Example

Three people X ,Y ,Z have been nominated for the Manager's post. The chances for getting elected for them are 0.4, 0.35 and 0.25 respectively. If X will be selected the probability that he will introduce Bonus scheme is 0.6 the respective chances in respect of Y and Z are 0.3 and 0.4 respectively. If it w known that Bonus scheme has been introduced, what is the probability that X is selected as a Manager ?

Solution

see probability-theory.pdf+++++ P47

CHAPITRE 3

RANDOM VARIABLES

Définition 3.0.3 *A random variable X (r.v.) is a measurable function from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the reals, i.e., it is a function*

$$X : \Omega \longrightarrow \mathbb{R}$$

such that $\forall B \in \mathcal{B}_{\mathbb{R}}$

$$X^{-1}(B) \in \mathcal{F}.$$

A random variable (r.v.) is defined as a function that associates a number to each element of the outcome space. Hence, any X ,

$$X : \Omega \longrightarrow \mathbb{R}$$

is a random variable.

Random variables are usually denoted by $X, Y, Z, \dots$.

3.1 Discrete random variables

As before, suppose Ω is a sample space.

Définition 3.1.1 A Random Variable X is said to be discrete if it takes only the values of the set $\{0, 1, 2, \dots\}$

Exemples 3.1.1 1. Tossing 2 coins simultaneously

$$\Omega = \{HH, HT, TH, TT\}$$

Let the random variable be getting number of heads then

$$X(\Omega) = \{0, 1, 2\}$$

2. Sum of the two numbers on throwing 2 dice

$$X(\Omega) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

3.1.1 Discrete Probability Distributions

The probability distribution of a discrete random variable X lists the values and their probabilities.

Value of X	x_1	x_1	x_1	$\dots$	x_k
Probability	p_1	p_2	p_3	$\dots$	p_k

where

$$0 \leq p_i \leq 1 \text{ and } p_1 + p_2 + \Delta\Delta\Delta + p_k = 1$$

We have

$$\mathbb{P}_X(x) = \mathbb{P}(X = x) = \mathbb{P}_X(X^{-1}(x))$$

Note that the pre-image $X^{-1}(x)$ is the event $\{\omega \in \Omega : X(\omega) = x\}$.

3.1.2 Cumulative Distribution Function (CDF) of a disc. r.v.

Définition 3.1.2 *The cumulative distribution function $F(x)$ of a discrete random variable X is*

$$F(x) = \mathbb{P}(X \leq x) = \sum \mathbb{P}(X = x)$$

3.2 Expectation

The expectation of a RV X is a real number computed by

$$E[X] = \sum_{k \in X(\Omega)} k \mathbb{P}(X = k).$$

if X is discrete and

$$E[X] = \int_{\mathbb{R}} x f(x) dx.$$

if X is continuous. Intuitively, the expectation is the ?average value? of the RV, more precisely it is a weighted average of the values of the RV, where the weights are the probabilities of the outcomes. The expectation is also called mean, expected value and sometimes average.

Expected Value Rule for Functions of Random Variables

Let X be a random variable with PMF p_X , and let $g(X)$ be a function of X . Then, the expected value of the random variable $g(X)$ is given by

$$E[g(X)] = \sum g(x) \mathbb{P}_X(x).$$

3.3 Variance

The variance $\text{var}(X)$ of a random variable X is defined by

$$\text{var}(X) = E[X^2] - (E[X])^2.$$

3.4 Continuous Random Variables

Définition 3.4.1 Random variable X is continuous if probability density function (pdf) f is continuous at all but a finite number of points and possesses the following properties :

- $f(x) \geq 0$, for all x ,
- $\int_{-\infty}^{+\infty} f(x)dx = 1$.

We have $\mathbb{P}(a < X \leq b) = \int_a^b f(x)dx$

The (cumulative) distribution function (cdf) for random variable X is

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^x f(t)dt,$$

and has properties

$$\lim_{x \rightarrow -\infty} F(x) = 0,$$

$$\lim_{x \rightarrow +\infty} F(x) = 1,$$

if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$; that is, F is nondecreasing,

$$\mathbb{P}(a < X \leq b) = \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a) = F(b) - F(a) = \int_a^b f(x)dx,$$

$$F'(x) = f(x)$$

CHAPITRE 4

IMPORTANT PROBABILITY DISTRIBUTIONS

4.1 Discrete probability distributions

4.1.1 The uniform distribution

If all the outcomes have equal probability, i.e $\mathbb{P}_1 = \mathbb{P}_2 = \dots = \mathbb{P}_n = \frac{1}{n}$, then the distribution is called a **uniform distribution**.

4.1.2 The Bernoulli distribution

4.1.3 The Binomial distribution (or Repeated Bernoulli Trials)

4.1.4 The geometric distribution

4.1.5 The Poisson distribution

4.1.6 The Bernoulli distribution

4.2 Discrete probability distributions

CHAPITRE 5

CHARACTERISTIC FUNCTIONS

Définition 5.0.1 *The characteristic function of a random variable X is the function $\varphi_x : \mathbb{R} \rightarrow \mathbb{C}$ defined by*

$$\varphi_x(t) = \mathbb{E}(e^{itX})$$

CHAPITRE 6

LIMIT THEOREMS

6.1 Markov and Chebyshev Inequalities

Théorème 6.1.1 (Markov Inequality) *If a random variable X can only take nonnegative values, then*

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$$

for all $a > 0$

Théorème 6.1.2 (Chebyshev Inequality) *If X is a random variable with mean μ and variance σ^2 , then*

$$\mathbb{P}(|X - \mu| \geq c) \leq \frac{\sigma^2}{C^2}$$

for all $c > 0$

6.2 The Weak Law of Large Numbers

Théorème 6.2.1 (The Weak Law of Large Numbers) *Let $X_1, X_2, \dots$ be independent identically distributed random variables with mean μ . For every $\epsilon > 0$, we have*

$$\mathbb{P}(|M_n - \mu| \geq \epsilon) = \mathbb{P}\left(\left|\frac{X_1 + X_2 + \dots + X_n}{n} - \mu\right| \geq \epsilon\right) \longrightarrow 0$$

as $n \longrightarrow \infty$

6.3 Convergence in Probability

see : d973b10c2587781f86ca4f2aff49098f P 40

6.4 The Central Limit Theorem

see : d973b10c2587781f86ca4f2aff49098f

CONCLUSION

Dans ce travail, on s'est intéressée à résoudre le problème d'un programme linéaire par la méthode de trajectoire centrale basée sur les fonctions noyaux, où on a présenté une nouvelle fonction noyau à terme barrière trigonométrique.

Nous avons analysé les versions à grande et petite mise à jour de l'algorithme primal-dual. Ces résultats sont des contributions importantes pour améliorer la complexité calculatoire du problème étudié.

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