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Algebra I: Course Notes.
Chapter 1. Notions of Logic.

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Life is good only for two things, discovering mathematics and teaching mathematics.

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Academic Year 2025/2026

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Chapter 1

Notions of Logic

Logic is the basis of mathematical reasoning. To solve problems and prove results, we need clear rules for working with assertions and their truth values. In this chapter, we learn what assertions are, how to use quantifiers to express statements about sets, and how to combine assertions with logical connectives such as "**and**," "**or**," "**not**," "**if. . . then**," and "**if and only if**." We also study the main methods of proof, including direct proof, proof by contradiction, proof by contrapositive, proof by cases, counterexamples, and mathematical induction. These tools will help us build precise arguments and understand mathematics more deeply.

1.1 Assertion and Truth Table

Definition 1

1. An assertion (sometimes called a proposition) in mathematics is a sentence P which is either true (denoted as T or 1) or false (denoted as F or 0) but not true and false at the same time.

2. T, F are called truth values of the assertion P. These values are usually displayed in a table called a truth table.

P	T	F

Truth table for P.

3. Assertions are usually denoted by capital letters such as P, Q, R...

Example 2

1. P: "The sum of two even integers is an even integer." This assertion is true T.

2. P: "$\sqrt{3}$ is a rational number". This assertion is false F, (since $\sqrt{3}$ cannot be expressed in the form $\frac{m}{n}$

with $m \in \mathbb{Z}, n \in \mathbb{Z}^*$).

3. P: "7 is a prime number". This assertion is true T.

1.2 Quantifiers and Quantified assertions

Quantifiers are symbols used to make assertions about elements of a set. There are two main types: the universal quantifier, which refers to all elements of a set, and the existential quantifier, which states that at least one element has a certain property. These quantifiers are important in logic and in mathematical proofs.

A quantified assertion is a logical assertion that uses quantifiers to describe properties of elements in a set. With them, we can say that something is true for every element (universal) or that it is true for at least one element (existential).

1.2.1 Quantifiers

Definition 3 ***(Quantifiers)***

*1. **Universal Quantifier**: The symbol $\forall$ is called the universal quantifier. It is usually read as "for all" , or " for every" , or "for each".*

*2. **Existential Quantifier**: The symbol $\exists$ is called the existential quantifier. It is usually read as "for some," "there exists," or "for at least one."*

*3. **Uniqueness quantifier:** The symbol $\exists!$ is called the uniqueness quantifier (or unique existential quantifier). It is usually read as "there exists exactly one," "there exists a unique," or "there exists one and only one."*

Example 4 *1. The assertion "The square of every real number is non-negative" can be expressed symbolically as*

$$\forall x \in \mathbb{R} : x^2 \geqslant 0.$$

2. The assertion "For every integer n, $2n$ is even" can be written as

$$\forall n \in \mathbb{Z} : 2n \text{ is even}.$$

3. The assertion "There exists an integer n such that $n^2 = 9$" can be expressed as

$$\exists n \in \mathbb{Z} : n^2 = 9.$$

4. The assertion "There exists exactly one natural number x such that $x - 2 = 4$" can be expressed as

$$\exists! x \in \mathbb{N} : x - 2 = 4.$$

1.2.2 Quantified assertions

Definition 5 ***(Quantified Assertions)***

1. An assertion that contains a universal or existential quantifier is called a ***quantified assertion****.*

2. Let S be a set and let $P(x)$ be a property defined for elements $x \in S$. Then

(a) A universal assertion (or universally quantified assertion) has the form

$$\forall x \in S : P(x).$$

It is true if and only if $P(x)$ holds for every $x \in S$. It is false if there exists at least one $x \in S$ for which $P(x)$ does not hold. Such an element is called a ***counterexample*** *to the universal assertion.*

(b) An existential assertion (or existentially quantified assertion) has the form

$$\exists x \in S : P(x).$$

It is true if there exists at least one $x \in S$ such that $P(x)$ holds. It is false if $P(x)$ is false for all $x \in S$.

(c) A uniqueness assertion (or uniquely quantified assertion) has the form

$$\exists! x \in S : P(x).$$

It is true if there exists exactly one element $x \in S$ such that $P(x)$ holds.

Remark 6 ***(Readings of Quantified Assertions)***

1. Universal assertion

$$\forall x \in S : P(x),$$

is read as "For all $x \in S$, the property $P(x)$ holds", "For every x, $P(x)$ holds" or "For each x, $P(x)$ holds".

2. Existential assertion

$$\exists x \in S : P(x),$$

is read as"There exists an $x \in S$ such that $P(x)$ holds" , "For some x, $P(x)$ holds".

3. Uniqueness assertion

$$\exists! x \in S : P(x),$$

is read as "There exists exactly one $x \in S$ such that $P(x)$ holds" or "There exists a unique $x \in S$ such that $P(x)$ holds"

Example 7 *Consider the following quantified assertions*

1.

$$\forall x \in \mathbb{R} : x^2 \geqslant 0.$$

This is a universal assertion, which states: "For all real numbers x, x^2 is greater than or equal to zero." This assertion is true, since the square of any real number is non-negative.

2.

$$\exists x \in \mathbb{Z} : x \textit{ is even}.$$

This is an existential assertion, which states: "There exists an integer x such that x is even." This assertion is true, as, for example, $x = 2$ is an even integer.

3.

$$\exists! x \in \mathbb{R} : x^2 = 1.$$

This is a uniqueness assertion, which states: "There exists exactly one real number x such that $x^2 = 1$." This assertion is false, because the equation $x^2 = 1$ admits two real solutions, namely $x = 1$ and $x = -1$, not exactly one.

1.3 Logical connectives

Logical connectives are operations that combine assertions to form new ones. They can be divided into two categories: **unary connectives**, which apply to a single assertion, and **binary connectives**, which connect two assertions.

The first symbol is a unary connective. It takes one assertion and produces another; for instance, from P it creates the assertion "*not* (P)." The remaining four symbols are binary connectives. Each of them requires two assertions in order to produce a third, even if the two given assertions are the same. The five standard connectives are: , $\wedge$, $\vee$, $\Longrightarrow$, and $\Longleftrightarrow$.

• The symbol is called **negation** and is read as "not."

• The symbol $\wedge$ is called **conjunction** and is read as "and."

• The symbol $\vee$ is called **disjunction** and is read as "or."

• The symbol $\Longrightarrow$ is called **implication** and is read as "implies."

• The symbol $\Longleftrightarrow$ is called **double implication**, or **equivalence**, and is read as "if and only if" or "is equivalent to."

1.3.1 Negation

Negation is a fundamental unary logical operator. It takes a single assertion and reverses its truth value: if the assertion is true, its negation is false; if the assertion is false, its negation is true.

Definition 8 ***(Negation)***

Let P be an assertion. The negation of P, denoted by $not(P)$ or $\bar{P}$, is the assertion that is true when P is false, and false when P is true. The truth table for the negation of P is given below

P	$not(P)$
T	F
F	T

Truth table for negation.

Example 9

1. If P: "6 is an odd number," then the negation is $not(P)$: "6 is not an odd number," which is equivalent to saying "6 is even".

2. If P: "x is an element of the set S," written as $x \in S$, then the negation is $not(P)$: "x is not an element of the set S," written symbolically as $x \notin S$.

*3. **Negation of quantified assertions.***

*(a) **Universal assertion.** The negation of a universal assertion is obtained by replacing $\forall$ with $\exists$ and negating the property. Then*

$$not(\forall x \in S : P(x))$$

is

$$\exists x \in S : \textit{not}\,(P(x))\,.$$

This means: "It is not true that every element of S satisfies $P(x)$; instead, there exists at least one element in S for which $P(x)$ does not hold."
For example, the negation of

$$\forall x \in \mathbb{R} : x^2 > x,$$

is

$$\exists x \in \mathbb{R} : \textit{not}(x^2 > x),$$

which can be rewritten as

$$\exists x \in \mathbb{R} : x^2 \leqslant x.$$

(b) ***Existential assertion.*** *The negation of an existential assertion is obtained by replacing $\exists$ with $\forall$ and negating the property. Then*

$$\textit{not}(\exists x \in S : P(x))$$

is

$$\forall x \in S : \textit{not}(P(x)).$$

This means: "It is not true that some element of S satisfies $P(x)$; instead, every element of S fails to satisfy $P(x)$."
For example the negation of

$$\exists x \in \mathbb{R} : x^2 = 2$$

is

$$\forall x \in \mathbb{R} : \textit{not}(x^2 = 2),$$

which can be rewritten as

$$\forall x \in \mathbb{R} : x^2 \neq 2.$$

Remark 10 *To negate a quantified assertion:*

- *Replace $\forall$ with $\exists$ (and vice versa).*
- *Negate the property $P(x)$.*
- *For the uniqueness quantifier ($\exists!$), express the negation as: "either no element satisfies $P(x)$, or more than one element satisfies it."*

1.3.2 Conjunction

In the previous section, we introduced a connector that applies to a single assertion. We now turn to a connector that applies to two assertions: the logical conjunction "and." The conjunction is a binary logical operator that links two assertions and expresses the idea that both are true at the same time.

Definition 11 *Let P and Q be two assertions. The conjunction of P and Q denoted by $P \wedge Q$ " read as P and Q" is the assertion that is true only when both P and Q are true, and false in all other cases. The truth table for $\wedge$ is shown below.*

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth table for Conjunction.

Example 12

1. Let P: "7 is a prime integer" and Q:"6 is an even integer". The conjunction is

$$P \wedge Q : \text{"7 is a prime integer and 6 is an even integer."}$$

This assertion is true, since both P and Q are true.

2. Let P: "$x > 1$" and Q: "$x < 3$". The conjunction is

$$P \wedge Q : \text{"}x > 1 \text{ and } x < 3.\text{"}$$

This means that x is greater than 1 and x less than 3. Then $P \wedge Q$:" x is in the open interval $]1, 3[$".

3. Let the assertions be P: "k is a multiple of 2" and Q: "k is a multiple of 3." Then the conjunction $P \wedge Q$ is:"k is a multiple of 2 and k is a multiple of 3," which is equivalent to saying "k is a multiple of 6."

Remark 13 ***(Truth Table Size)***

The number of rows in a truth table depends on the number of component assertions:

(a) For a compound assertion with one component (a single assertion), the truth table contains 2 possibilities: T or F.

(b) For a compound assertion with two components (two assertions), the truth table contains $2^2 = 4$ *possibilities.*

(c) In general, for a compound assertion with n components (i.e., n assertions), the truth table contains 2^n *possibilities.*

1.3.3 Disjunction

We now introduce the counterpart of conjunction, namely the logical disjunction "or." Disjunction is a binary logical operator that links two assertions and expresses the idea that at least one of them is true.

Definition 14 ***(Disjunction)*** *Let P and Q be two assertions. The disjunction of P and Q, denoted by* $P \vee Q$ *"read as P or Q", is the assertion that is true whenever at least one of P or Q is true, and false only when both P and Q are false. The truth table of the operator* $\vee$ *is therefore given as follows*

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth table for Disjunction.

Example 15

1. *The assertion "*$\pi \geqslant 2$ *or* $e \geqslant 1$*" is true, since both inequalities are satisfied.*

2. *Let P: "*$x < 2$*" and Q: "*$x > 10$*."Then the disjunction* $P \vee Q$ *is equivalent to "*$x \in]-\infty, 2[\cup]10, +\infty[$*."*

3. *Let P: "n is a multiple of* 3 *less than* 10*" and Q: "n is an even number less than* 10*." Then the assertion* $P \vee Q$ *corresponds to the set* $\{0, 2, 3, 4, 6, 8, 9\}$*.*

4. *Let P: "* $\sqrt{2}$ *is a rational number" and Q: "*2 *is an odd number," then the disjunction* $P \vee Q$ *can be expressed as "* $\sqrt{2}$ *is rational or* 2 *is odd."*

5. *Let P: "x is an element of the set* S_1*" and Q: "x is an element of the set* S_2*," then* $P \vee Q$ *is equivalent to "*$x \in S_1 \cup S_2$*" (the union of* S_1 *and* S_2*).*

6. *The assertion "*$2 \leqslant 2$*" means "*2 *is less than* 2 *or* 2 *equals* 2*." It is true because* $2 = 2$*.*

1.3.4 Logical Implication

Logical implication is a derived logical operator, constructed from the fundamental operators "not" and "or". It is the foundation of most mathematical reasoning.

Definition 16 *(**Logical Implication**) Let P and Q be two assertions.*

*1. The implication of Q by P, denoted $(P \Longrightarrow Q)$ (read as "P implies Q"), is the assertion that is false **only** when P is true and Q is false. In all other cases, the implication is true. The truth table of logical implication is*

P	Q	$P \Longrightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth table for Implication.

*2. The symbol $\Longrightarrow$ is called the **implication arrow** and represents the direction of the logical implication.*

*3. The logical implication can be expressed using the operators "**not**" and "**or**"*

$$not(P) \vee Q.$$

*4. P is called the **hypothesis**, and Q is called the **conclusion** (or **consequence**).*

*5. The assertion $(Q \Longrightarrow P)$ is called the **converse** of $(P \Longrightarrow Q)$.*

*6. The assertion $(not(Q) \Longrightarrow not(P))$ is called the **contrapositive** of $(P \Longrightarrow Q)$. The contrapositive always has the same truth value as the original implication.*

Example 17

1. Let P: "n is divisible by 4," and Q: "n is even.". Then

(a) Implication $(P \Longrightarrow Q)$: "If n is divisible by 4, then n is even." is True.

(b) Converse $(Q \Longrightarrow P)$: "If n is even, then n is divisible by 4." is not always true.

(c) Contrapositive $(not(Q) \Longrightarrow not(P))$:"If n is not even, then n is not divisible by 4." is true.

2. Let P: "$x > 5$," and Q: "$x > 0$." Then

(a) Implication $(P \Longrightarrow Q)$: "If $x > 5$, then $x > 0$." is true.

(b) Converse $(Q \Longrightarrow P)$: "If $x > 0$, then $x > 5$." is not always true.

(c) Contrapositive $(not(Q) \Longrightarrow not(P))$: "If $x \leqslant 0$, then $x \leqslant 5$." is true

3. The assertion " if $x \leqslant 1$ then $x^2 \leqslant 1$" (is false, for example, if $x = -2$, then $x \leqslant 1$ is true, but $x^2 = 4$, which is not less than or equal to 1) can be written as the implication

$$x \leqslant 1 \Longrightarrow x^2 \leqslant 1.$$

The converse is

$$x^2 \leqslant 1 \Longrightarrow x \leqslant 1.$$

The Contrapositive is

$$\textit{not}\left(x^2 \leqslant 1\right) \Longrightarrow \textit{not}\,(x \leqslant 1)$$

this is equivalent to

$$x^2 > 1 \Longrightarrow x > 1.$$

This means, if the square of x is greater than 1, then x is greater than 1.

4. Let x, y be two real numbers, then the contrapositive of the implication

$$xy = 0 \Longrightarrow x = 0 \vee y = 0,$$

is

$$\textit{not}\,(x = 0 \vee y = 0) \Longrightarrow \textit{not}(xy = 0),$$

which simplifies to

$$x \neq 0 \wedge y \neq 0 \Longrightarrow xy \neq 0.$$

1.3.5 Logical Equivalence

The logical equivalence connector, denoted $\Longleftrightarrow$, links two assertions and indicates that they are either both true or both false. In other words, in every possible situation, the two assertions have the same truth value.

Definition 18 ***(Logical Equivalence or double implication)***

1. Two assertions P and Q are said to be logically equivalent if they have the same truth value in all cases, that is, their truth tables are identical.

2. The logical equivalence of P and Q is denoted by $P \Longleftrightarrow Q$. This relation can be expressed as a combination of two implications $(P \Longrightarrow Q)$ and $(Q \Longrightarrow P)$, which is known as a double implication.

3. The assertions $P \Longleftrightarrow Q$ is true if P and Q are both true or both are false. Otherwise, it is false.

The truth table for $\Longleftrightarrow$ is shown below.

P	Q	$P \Longleftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth table for equivalence.

Remark 19 *The assertion $P \Longleftrightarrow Q$ can be expressed in several equivalent ways:*

- *P is equivalent to Q.*
- *P implies Q and Q implies P.*
- *P is true if and only if Q is true.*
- *P is true iff Q is true (where "iff" stands for "if and only if").*
- *For P to be true, it is necessary and sufficient that Q is true.*
- *P is a necessary and sufficient condition (NSC) for Q.*

Example 20

1. For two real numbers x and y, the equivalence

$$x \cdot y = 0 \Longleftrightarrow x = 0 \vee y = 0,$$

is true.

2. For two real numbers x and y, the equivalence

$$x \cdot y = 0 \Longleftrightarrow x = 0 \wedge y = 0,$$

is false.

3. Let $x \in \mathbb{R}$. The assertion "$x \in [0,1]$" is not equivalent to "$x \in \mathbb{R}_+$". Indeed, the direct implication

$$x \in [0,1] \Longrightarrow x \in \mathbb{R}_+,$$

is true, but the converse implication

$$x \in \mathbb{R}_+ \Longrightarrow x \in [0,1],$$

is false.

4. *Let $x \in \mathbb{R}$. The assertion*

$$x^2 = 1 \Leftrightarrow x = +1 \vee x = -1,$$

is true, because both implications hold.

Proposition 21 ***(Basic Logical Laws)***

Let P, Q, and R be three assertions, then the following equivalences hold:

1. *The double negation*

$$not(not(P)) \Longleftrightarrow P.$$

2. *Commutativity of Conjunction and Disjunction*

$$P \wedge Q \Longleftrightarrow Q \wedge P, P \vee Q \Longleftrightarrow Q \vee P.$$

3. *Associativity of Conjunction and Disjunction*

$$\begin{aligned} (P \wedge Q) \wedge R &\Longleftrightarrow P \wedge (Q \wedge R) \\ (P \vee Q) \vee R &\Longleftrightarrow P \vee (Q \vee R). \end{aligned}$$

4. *Distributivity of Conjunction over Disjunction and vice versa*

$$\begin{aligned} P \wedge (Q \vee R) &\Longleftrightarrow (P \wedge Q) \vee (P \wedge R) \\ P \vee (Q \wedge R) &\Longleftrightarrow (P \vee Q) \wedge (P \vee R). \end{aligned}$$

5. *The contrapositive law*

$$(P \Longrightarrow Q) \Longleftrightarrow (not(Q) \Longrightarrow not(P)).$$

6. ***De Morgan's Laws:*** *The negation of conjunction and disjunction (negation is distributive over conjunction and disjunction)*

$$\overline{(P \wedge Q)} \Longleftrightarrow \overline{P} \vee \overline{Q}, \qquad \overline{(P \vee Q)} \Longleftrightarrow \overline{P} \wedge \overline{Q}.$$

Proof. Verify these equivalences by constructing truth tables for each law. ■

Conclusion 22 *The table below summarizes the truth values of the main logical connectors introduced so far. True (resp. False) is represented by T (resp. F):*

P	Q	$\bar{P}$	$P \wedge Q$	$P \vee Q$	$P \Longrightarrow Q$	$P \Longleftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

1.4 Methods of Proof

Mathematical proof methods are formal processes used to demonstrate truths or establish conclusions based on axioms, theorems, and logical rules. There are several commonly used methods of proof, including: Direct Proof, Proof by Contradiction, Proof by Contrapositive, Proof by Cases, Proof by Counterexample and Mathematical induction.

Definition 23 ***(Definition of a proof)***
A proof is a sequence of logical steps that leads from given or known assertions (called assumptions) to a final assertion (called the conclusion), where each step follows by a valid logical implication.

1.4.1 Direct Proof

The direct method is a proof technique in which one starts from the given hypotheses and proceeds step by step to reach a conclusion through a sequence of logical implications.

Definition 24 ***Direct Proof (Principle).*** *Let P and Q be two assertions. To prove the implication ($P \Longrightarrow Q$) directly, we assume that P is true and then show, through a sequence of logical implications that Q must also be true. The proof begins with "Assume P is true" and concludes with "Therefore, Q is true".*

Example 25

1. ***Assertion:*** *If r and s are rational numbers, then $r + s$ is rational. Formally, this can be written as*

$$r, s \in \mathbb{Q} \Longrightarrow r + s \in \mathbb{Q}.$$

Solution:

Assume that r and s are rational. By definition, $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers a, b, c, d with $b \neq 0$ and $d \neq 0$. Then

$$r + s = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Since $ad + bc$ and bd are integers, $r + s$ is a quotient of integers and therefore rational ($r + s \in \mathbb{Q}$).

2. ***Assertion:*** *Let x be an integer. If x is odd, then $x + 1$ is even. This can be written as*

$$x \in \mathbb{Z} : x \text{ is odd } \implies x + 1 \text{ is even.}$$

Solution:

Assume x is odd. Then $x = 2k + 1$ for some integer k. It follows that

$$x + 1 = 2k + 1 + 1 = 2(k + 1).$$

Since $k + 1$ is an integer, $x + 1$ is even.

3. ***Assertion:*** *For all integers m and n, if m and n are odd, then $m + n$ is even. This can be written as*

$$\forall m, n \in \mathbb{Z} : m \text{ is odd } \wedge \; n \text{ is odd } \implies m + n \text{ is even.}$$

Solution:

Assume m and n are odd integers. Then $m = 2k_1 + 1$ and $n = 2k_2 + 1$ for integers k_1, k_2. Therefore

$$m + n = (2k_1 + 1) + (2k_2 + 1) = 2(k_1 + k_2 + 1) = 2k_3,$$

where $k_3 = k_1 + k_2 + 1 \in \mathbb{Z}$. Hence, $m + n$ is even.

4. ***Assertion:*** *If n is an odd integer, then n^2 is odd. This can be written as*

$$n \in \mathbb{Z} : n \text{ is odd } \implies n^2 \text{ is odd.}$$

Solution:

Assume n is odd. Then $n = 2k + 1$ for some integer k. It follows that

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1,$$

where $m = 2k^2 + 2k \in \mathbb{Z}$. *Therefore,* n^2 *is odd.*

1.4.2 Proof by Contrapositive

Contraposition, or proof by contrapositive, is a method of proof that involves establishing the implication "if $non(Q)$, then $non(P)$" from the original implication "if P, then Q." The assertion "if $non(Q)$, then $non(P)$" is called the contrapositive of "if P, then Q." This method is particularly useful when proving $P \implies Q$ directly is difficult, but proving its contrapositive $(not(Q) \implies not(P))$ is simpler. It relies on the logical equivalence between an implication and its contrapositive.

Definition 26 ***Proof by Contrapositive (Principle)***. *Proof by contrapositive is based on the fact that any implication is logically equivalent to its contrapositive. For any two assertions P and Q*

$$(P \implies Q) \iff (not(Q) \implies not(P)).$$

Instead of proving $(P \implies Q)$ *directly, we prove its contrapositive* $(not(Q) \implies not(P))$. *This involves assuming* $not(Q)$ *is true and deducing that* $not(P)$ *must also be true.*

Example 27 *Let* $n \in \mathbb{Z}$ *be an integer. Provide a proof by contrapositive of the assertion."If* n^2 *is an odd integer, then n is odd." Formally, this can be written as*

$$n \in \mathbb{Z} : n^2 \textit{ is odd } \implies n \textit{ is odd.}$$

Solution: *The contrapositive of this assertion is "If n is not odd, then* n^2 *is not odd." In other words, "if n is even, then* n^2 *must also be even". Formally*

$$n \in \mathbb{Z} : n \textit{ is even} \implies n^2 \textit{ is even .}$$

Assume n is even. Then there exists an integer k such that

$$n = 2k.$$

Squaring n gives

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2) = 2k', k' = 2k^2 \in \mathbb{Z},$$

so n^2 is even. Since the contrapositive is true, the original assertion is also true: if n^2 is odd, then n must be odd.

1.4.3 Proof by Contradiction

Proof by contradiction, is a method of proof in which we assume that the assertion we want to prove is false, and then show that this assumption leads to a logical contradiction. Once a contradiction is reached, the assumption is rejected, and the original assertion is concluded to be true. This method has some similarities with proof by contrapositive, but it is not quite the same. In contrapositive proofs, we reformulate the assertion into an equivalent one; in contradiction proofs, we assume the opposite of the desired result and demonstrate that such an assumption is impossible.

Definition 28 ***Proof by Contradiction (Principle).*** *To prove that an assertion P is true, we assume the opposite (This means that P is false) and then show that this assumption leads to a contradiction or a false result. Since the assumption cannot hold, it follows that P must be true.*

Example 29

1. Prove that $\sqrt{2}$ is irrational. The classical proof of the irrationality of $\sqrt{2}$ is by contradiction.

Solution. *Assume, for contradiction, that $\sqrt{2}$ is rational. Then there exist integers $m, n \in \mathbb{N}$, with $n \neq 0$ and m and n coprime (i.e., having no common divisor other than 1), such that*

$$\sqrt{2} = \frac{m}{n}.$$

Squaring both sides gives

$$m^2 = 2n^2,$$

so m^2 is even. By the previous example on contrapositive, this implies that m is even. Hence, there exists $k \in \mathbb{N}$ such that $m = 2k$. Substituting, we obtain

$$n^2 = 2k^2,$$

which shows that n^2 is even, and therefore n is even. Thus, both m and n are divisible by 2, contradicting the assumption that they are coprime. Therefore, $\sqrt{2}$ cannot be rational. Hence, $\sqrt{2}$ is irrational. This completes the proof by contradiction.

2. Let $a, b > 0$ be two positive real numbers. Show that if

$$\frac{a}{1+b} = \frac{b}{1+a},$$

then

$$a = b.$$

Solution. *Suppose, for contradiction, that*

$$\frac{a}{1+b} = \frac{b}{1+a}$$

and

$$a \neq b.$$

Then

$$\frac{a}{1+b} = \frac{b}{1+a} \Longrightarrow a(a+1) = b(b+1),$$

so

$$a^2 + a = b^2 + b \Longrightarrow a^2 - b^2 = b - a,$$

which factors as

$$(a-b)(a+b) = -(a-b).$$

Since $a \neq b$, we can divide both sides by $a - b$,

$$a + b = -1.$$

But this is impossible, since a and b are positive real numbers and their sum must also be positive. This contradiction shows that our assumption was false. Therefore, it must be the case that $a = b$.

Conclusion. *If $\dfrac{a}{1+b} = \dfrac{b}{1+a}$, then necessarily $a = b$.*

1.4.4 Proof by Cases

When proving an assertion, it is sometimes easier to divide the argument into particular subcases. If these subcases together cover all possible situations, then the assertion is proved in complete generality. This approach is called proof by cases. In this method, the assertion to be proved is broken down into a finite number of cases (sub-assertions), each established independently.

Definition 30 ***Proof by Cases (Principle).*** *To prove $(P \implies Q)$, we divide the assumption P into several possible cases and show that the implication holds for each case. If P can be expressed as*

$$P \iff A_1 \vee A_2 \vee \cdots \vee A_n,$$

where each A_i represents a possible case. To prove $(P \implies Q)$, it suffices to show (the individual implications)

$$A_1 \implies Q, A_2 \implies Q, \ldots, A_n \implies Q.$$

If all these implications are true, it follows that

$$A_1 \vee A_2 \vee \cdots \vee A_n \implies Q,$$

and therefore

$$P \implies Q.$$

Example 31

1. Prove the assertion: if n is an integer, then $3n^2 + n + 14$ is even. Formally, this can be written as

$$n \in \mathbb{Z} \implies 3n^2 + n + 14 \text{ is even.}$$

Solution: *Let $n \in \mathbb{Z}$ be an integer number. We will consider two cases: when n is even and when n is odd.*

Case 1. *Suppose n is even. Then there exists $k \in \mathbb{Z}$ such that $n = 2k$, we get*

$$3n^2 + n + 14 = 3(2k)^2 + 2k + 14 = 12k^2 + 2k + 14 = 2(6k^2 + k + 7).$$

Setting

$$m_1 = 6k^2 + k + 7 \in \mathbb{Z},$$

we obtain

$$3n^2 + n + 14 = 2m_1.$$

Hence $3n^2 + n + 14$ is even when n is even.

Case 2. *Suppose n is odd. Then there exists $k \in \mathbb{Z}$ such that $n = 2k+1$, we get*

$$3n^2 + n + 14 = 3(2k+1)^2 + (2k+1) + 14 = 12k^2 + 14k + 18 = 2(6k^2 + 7k + 9).$$

Setting

$$m_2 = 6k^2 + 7k + 9 \in \mathbb{Z},$$

we have

$$3n^2 + n + 14 = 2m_2.$$

Thus the expression is also even when n is odd. Since in both cases $3n^2 + n + 14$ is even, it follows that if n is an integer, then $3n^2 + n + 14$ is even.

2. Prove the following assertion: If x is a real number, then $|x+3| - x > 2$. Formally, this can be written as

$$x \in \mathbb{R} \implies |x+3| - x > 2.$$

Solution: *Let $x \in \mathbb{R}$. Since the absolute value depends on the sign of $x+3$, we have*

$$|x+3| = \begin{cases} x+3, \text{ if } x \geqslant -3 \\ -x-3, \text{ if } x < -3. \end{cases}$$

Then, we consider two cases:

Case 1: *if $x \geqslant -3$, then $|x+3| - x = 3 > 2$, so the inequality holds.*

Case 2: *if $x < -3$, then $|x+3| - x = -2x - 3$. Thus*

$$-2x - 3 > 2.3 - 3 = 3 > 2.$$

Again, the inequality holds. Since the assertion is true in both cases, we conclude that for all real numbers x,

$$|x+3| - x > 2.$$

1.4.5 Proof by Counterexample

A counterexample is a specific and concrete example that demonstrates the falsity of a general claim, statement, conjecture, rule, or law. To disprove an assertion, it is sufficient to find just one counterexample showing that the claim does not always hold. In many cases, constructing an appropriate counterexample can be as challenging as proving that an assertion is true.

Definition 32 ***Proof by Counterexample (Principle)***. *Let $P(x)$ be a property defined for elements x in a set S. To show that the universal assertion*

$$\forall x \in S : P(x),$$

is false, it is sufficient to find an element $x \in S$ for which $P(x)$ does not hold. Such an element is called a counterexample to the assertion.

Example 33 *1. Show that the following assertion is false: "Every positive integer is the sum of three squares." Formally*

$$\forall x \in \mathbb{N}, \exists a, b, c \in \mathbb{N} : x = a^2 + b^2 + c^2.$$

The property is

$$P(x) : \exists a, b, c \in \mathbb{N} : x = a^2 + b^2 + c^2.$$

For example,

$$14 = 1^2 + 2^2 + 3^2.$$

So $P(14)$ is true.

Solution. *A counterexample is the integer 7. The possible squares less than or equal to 7 are $0, 1^2 = 1, 2^2 = 4$. However, for any $a, b, c \in \{0, 1, 2\}$, the sum $a^2 + b^2 + c^2$ cannot equal 7 (since $0 + 1 + 4 = 5$, not 7). Thus $P(7)$ is false, so 7 is a counterexample. Therefore, the universal assertion is false.*

2. Consider the assertion "For all real numbers x, we have $x^2 > x$." Formally

$$\forall x \in \mathbb{R} : x^2 > x.$$

Find a counterexample to show that this assertion is false.

Solution. *Consider $x = \frac{1}{2}$. Then*

$$x^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2} = x.$$

Since we have found a real number x for which $x^2 < x$, the universal assertion is false.

1.4.6 Proof by Mathematical Induction

)ÿ+L Proof by mathematical induction is a technique used to establish the truth of assertions or propositions for all natural numbers or for integers within a specified range. The process consists of two main steps: the base case and the inductive step.

Definition 34 ***(The Principle of Induction)*** *Let n_0 be a positive integer ($n_0 \in \mathbb{N}$), and let $\mathcal{P}(n)$ be an assertion about n for each positive integer $n \geqslant n_0$. To prove that $\mathcal{P}(n)$ is true for all $n \geqslant n_0$, the method of mathematical induction consists of two steps:*
(a) ***Base case:*** *Verify that $\mathcal{P}(n_0)$ is true.*
(b) ***Inductive step:*** *Assume that $\mathcal{P}(k)$ is true for some positive integer $k \geqslant n_0$ (this assumption is called the induction hypothesis). Then prove that $\mathcal{P}(k+1)$ is also true.*
If both steps are verified, it follows by induction that $\mathcal{P}(n)$ holds for all integers $n \geqslant n_0$.

Example 35 *Prove by induction that for all integers $n \geqslant 1$, the sum of the first n integers is given by the formula*

$$1 + 2 + ... + n = \frac{n(n+1)}{2}.$$

Solution: *Let $\mathcal{P}(n)$ be the assertion*

$$1 + 2 + ... + n = \frac{n(n+1)}{2}.$$

We will prove by induction that $\mathcal{P}(n)$ holds for all $n \geqslant 1$.
Base case: *If $n = 1$, then the assertion becomes $1 = \frac{1(1+1)}{2}$. The base case $\mathcal{P}(1)$ holds.*
Induction Hypothesis: *Assume $\mathcal{P}(k)$ is true for some integer $k \geqslant 1$, i.e.*

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Inductive Step: *We need to prove that $P(k+1)$ holds, i.e.*

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}.$$

Using the induction hypothesis,

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1).$$

Simplify

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

This is exactly the desired formula. Therefore, $\mathcal{P}(k+1)$ is true.

Conclusion: *By the Principle of Mathematical Induction, the assertion $\mathcal{P}(n)$ holds for all integers $n \geqslant 1$.*

Conclusion of the Chapter

In this chapter, we introduced the foundations of mathematical logic and explored several methods of proof. We began by studying logical assertions, connectives, truth tables, and fundamental logical laws, which provide the essential tools for rigorous reasoning.

We then examined key proof techniques, including direct proof, proof by contrapositive, proof by contradiction, proof by cases, proof by counterexample, and mathematical induction. Each method serves a specific purpose: direct proofs establish implications step by step; contrapositives and contradictions leverage logical equivalence and refutation; counterexamples demonstrate the limits of general statements; and induction allows us to prove assertions about infinite sets, such as the natural numbers.

Together, these concepts form the core of mathematical reasoning. They ensure the validity of results while fostering the rigor and clarity that are essential for advanced studies in mathematics and computer science.