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Numerical Series

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Chapter 1

Numerical series

1.1 Series with real or complex terms

Definition 1.1.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real or complex numbers. We call a numerical series (respectively complex series) of general term u_n, any expression of the form:*

$$u_0 + u_1 + ... + u_n + ... = \sum_{n\geq 0} u_n. \tag{1.1}$$

The real numbers (respectively the complexe numbers) $u_0, u_1, ...\ u_n, ...$ are called terms of the series.
Let us now consider the following partial sums:

$$S_n = u_0 + u_1 + ... + u_n = \sum_{k=0}^{n} u_k. \tag{1.2}$$

The number S_n is called partial sum of order n of the series $\sum\limits_{n\geq 0} u_n$, and the sequence (S_n) is called sequence of partial sums of the series $\sum\limits_{n\geq 0} u_n$.

Definition 1.1.2. *Let $\sum\limits_{n\geq 0} u_n$ be a series with real terms or complex. We say that the series $\sum\limits_{n\geq 0} u_n$ converges if the sequence of partial sums (S_n) converges, and it diverges if the sequence of partial sums diverges.*

Definition 1.1.3. *When the series* $\sum\limits_{n\geq 0} u_n$ *converges, we call sum of the series, the limit* S *of the sequence of partial sums and we write:*

$$S = \lim_{n\to+\infty} S_n = \sum_{n\geq 0} u_n. \tag{1.3}$$

Definition 1.1.4. *Let* $\sum\limits_{n\geq 0} u_n$ *be a convergent series of sum* S*. We call rest of order* n *of the series* $\sum\limits_{n\geq 0} u_n$*, the number* R_n *which defined by:*

$$R_n = S - S_n = \sum_{k\geq n+1} u_k. \tag{1.4}$$

We then have the following equivalence:

$$\sum_{n\geq 0} u_n \text{ converge } \Leftrightarrow \lim_{n\to+\infty} S_n = S \Leftrightarrow \lim_{n\to+\infty} R_n = 0. \tag{1.5}$$

Remak 1.1. *We also deduce that the nature of a series does not change, in removing a finite number of its terms. On the other hand, if the series converges, the value of its sum depends on all the terms of the series.*

Example 1.1.1. *(Geometric series)*
The geometric series $\sum\limits_{n\geq 0} R^n$ *is:*

1. *convergent if and only if* $|R| < 1$*, in this case* $S = \dfrac{1}{1-R}$.
2. *divergent if and only if* $|R| \geq 1$.

Proposition 1.1.1. *(* ***Telescopic process****) Let* (u_n) *and* (v_n) *be two sequences of real or complex numbers, such that* $u_n = v_{n+1} - v_n$*. Then, the series* $\sum\limits_{n\geq 0} u_n$ *converges if and only if the sequence* (v_n) *converges, and in this case:*

$$\sum_{n\geq 0} u_n = \lim_{n\to+\infty} v_n - v_0. \tag{1.6}$$

Proof. Indeed, the proof is made of the following equality:

$$\sum_{k=0}^{n} u_k = \sum_{k=0}^{n} (v_{k+1} - v_k) = v_{k+1} - v_0.$$

□

Example 1.1.2. *(Case of convergence) Let us consider the series of general term*

$$u_n = \frac{1}{n(n+1)}, \; n \geq 1.$$

The term u_n can be rewritten as:

$$u_n = \frac{1}{n} - \frac{1}{n+1}, n \geq 1.$$

Therefore:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} u_k = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Since $\lim_{n\to+\infty} S_n = 1$, it follows that the series $\sum_{n\geq 1} u_n$ is convergent of sum $S = 1$.

Example 1.1.3. *(Case of divergence) Let us consider the series of general term*

$$u_n = \ln(1 + \frac{1}{n}), \; n \geq 1.$$

The term u_n can be rewritten as:

$$u_n = \ln(n+1) - \ln(n), \; n \geq 1. \tag{1.7}$$

Hence:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} u_k = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \dots + (\ln(n+1) - \ln(n)) \\ &= \ln(n+1). \end{aligned}$$

Since $\lim_{n\to+\infty} S_n = +\infty$, it follows that the series $\sum_{n\geq 1} u_n$ diverges.

Proposition 1.1.2. *(Necessary condition of convergence)* *For a numerical series $\sum_{n\geq 0} u_n$ to be convergent, it is necessary that its general term u_n tends towards zero.*

Proof. Suppose that $\sum_{n\geq 0} u_n$ converges to $S = \lim_{n\to+\infty} S_n$. We then have:

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} S_n - S_{n-1} = S - S = 0. \tag{1.8}$$

Corollary 1.1.1. *(Sufficient condition of divergence) A sufficient condition for a serie is divergent, is that its general term does not tend towards zero.*

Corollary 1.1.2. *The converse of Proposition 1.1.2 is false in general. Indeed, the series* ***harmonic*** $\sum_{n\geq 1}\frac{1}{n}$ *diverges, while its general term tends towards zero.*

□

1.1.1 Algebraic structure of the set of convergent series

Proposition 1.1.3. *Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ be two numerical or complex series. We then have the following properties:*

1. If $\sum_{n\geq 0} u_n$ is convergent with sum S_1 and if $\sum_{n\geq 0} v_n$ is convergent with sum S_2, then $\sum_{n\geq 0}(u_n + v_n)$ is convergent with sum $S_1 + S_2$.

2. If $\sum_{n\geq 0} u_n$ is convergent with sum S_1 and if $\alpha \in \mathbb{R}$ (where $\mathbb{C}$), then $\sum_{n\geq 0}(\alpha u_n)$ is convergent with sum αS_1.

Proof. The proof of this proposition follows immediately from the properties of the limits of sequences. □

Remak 1.2. *With these two previous operations, we can easily demonstrate that the set of convergent series is a subspace vector.*

1.1.2 Other algebraic operations

Proposition 1.1.4. *Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ be two numerical or complex series. We then have the following supplimentary properties:*

1. If $\sum_{n\geq 0} u_n$ diverges and if $\alpha \in \mathbb{R}^*$, then $\sum_{n\geq 0}(\alpha u_n)$ diverges.

2. If $\sum_{n\geq 0} u_n$ converges and if $\sum_{n\geq 0} v_n$ diverges, then $\sum_{n\geq 0}(u_n + v_n)$ diverges.

3. If the two series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ are divergent, we cannot conclude anything about the nature of the series $\sum_{n\geq 0}(u_n + v_n)$, it can be convergent, as it can be divergent.

Proof. The proof of this proposition follows immediately from the properties of the limits of sequences. □

1.1.3 Cauchy criterion

Theorem 1.1.1. *Let $\sum\limits_{n\geq 0} u_n$ be a series with real terms or complex. This series is convergent if and only if:*

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall p, q \in \mathbb{N},\ p > q \geq n_0,\ \text{we have } \left| \sum_{k=q+1}^{p} u_k \right| < \epsilon. \tag{1.9}$$

Proof. The proof of this theorem is done using the Cauchy criterion following the partial sums $S_n = \sum\limits_{k=0}^{n} u_k$, and the fact that $S_p - S_q = \sum\limits_{k=q+1}^{p} u_k$. □

1.2 Positive terms series

Definition 1.2.1. *We call a series with a positive terms any series whose general term u_n verifies:*

$$u_n \geq 0, \text{ for all } n \geq 0. \tag{1.10}$$

Proposition 1.2.1. *Let $\sum\limits_{n\geq 0} u_n$ be a series with real terms positive. Then this series converges towards S, if and only if the sequence (S_n) of its partial sums is majorized. In this case, we have:*

$$S_n \leq S, \text{ for all } n \geq 0. \tag{1.11}$$

Proof. Since $S_{n+1} - S_n = u_{n+1} \geq 0$, for all $n \geq 0$, it follows that (S_n) is increased, so for it to be convergent, it is necessary and sufficient that it be majorized. In this case, the limit of the sequence of partial sums (S_n) majore all the terms of the sequence. □

Remak 1.3. *If (S_n) is not majorized, $\lim_{n\to+\infty} S_n = +\infty$, and the series $\sum\limits_{n\geq 0} u_n$ diverges.*

1.2.1 Comparison theorems

Theorem 1.2.1. *Given two series with positive terms* $\sum\limits_{n\geq 0} u_n$ *and* $\sum\limits_{n\geq 0} v_n$ *verifying :*

$$\exists n_0 \in \mathbb{N}, \text{ such that for all } n \geq n_0, u_n \leq v_n, \tag{1.12}$$

we then have:

1. $\sum\limits_{n\geq 0} v_n$ *converge implies* $\sum\limits_{n\geq 0} u_n$ *converges.*
2. $\sum\limits_{n\geq 0} u_n$ *diverges implies* $\sum\limits_{n\geq 0} v_n$ *diverges.*

Proof. 1. Let's pose for all $n \in \mathbb{N}$, $S_n = \sum\limits_{k=0}^{n} u_k$ and $T_n = \sum\limits_{k=0}^{n} v_k$. Since $u_n \leq v_n$, for all $n \geq n_0$, we then obtain :

$$S_n \leq T_n, \text{ for all } n \geq n_0. \tag{1.13}$$

If the series $\sum\limits_{n\geq 0} v_n$ converges, the sequence (T_n) is therefore majorized, then the sequence (S_n) is also majorizd, and thus the series $\sum\limits_{n\geq 0} u_n$ converges.
2. The second property is the contrapositive of the first, so is also true. □

Corollary 1.2.1. *Let* $\sum\limits_{n\geq 0} u_n$ *and* $\sum\limits_{n\geq 0} v_n$ *be two series with positive terms. Suppose that there exist two strictly positive real numbers* α *and* β *verifying:*

$$\alpha u_n \leq v_n \leq \beta u_n, \tag{1.14}$$

then $\sum\limits_{n\geq 0} u_n$ *and* $\sum\limits_{n\geq 0} v_n$ *are of the same nature.*

Proof. Applying the Theorem 1.2.1 twice, gives us: if the series with general term v_n converges, the series with general term u_n converges and if the series with general term u_n converges, the series with general term v_n converges. Which shows that the two series are of the same nature. □

Example 1.2.1. *Consider the general term number series:*

$$u_n = \frac{\theta^n}{\sqrt{n}}, \ \theta \geq 0 \text{ and } n > 0. \tag{1.15}$$

* *If* $\theta \geq 1$, $u_n \geq \frac{1}{\sqrt{n}} \geq \frac{1}{n}$. *Since* $\sum\limits_{n\geq 1} \frac{1}{n}$ *diverges, the considered series also diverges.*
* *If* $0 \leq \theta < 1$, $u_n \leq \theta^n$. *Since* $\sum\limits_{n\geq 1} \theta^n$ *converges (geometric series), the considered series also converges.*

Theorem 1.2.2. *Let $\sum\limits_{n\geq 0} u_n$ and $\sum\limits_{n\geq 0} v_n$ be two series with positive terms. Suppose there exists a positive real l (or $l = +\infty$), such that $\lim_{n\to+\infty} \dfrac{u_n}{v_n} = l$, we then have:*
1. If $l = 0$ and the series $\sum\limits_{n\geq 0} v_n$ converge, the series $\sum\limits_{n\geq 0} u_n$ converges.
2. If $l = +\infty$ and the series $\sum\limits_{n\geq 0} v_n$ diverge, the series $\sum\limits_{n\geq 0} u_n$ diverges.
3. If $l \neq 0$ and $\neq +\infty$, both $\sum\limits_{n\geq 0} v_n$ and $\sum\limits_{n\geq 0} u_n$ are of the same nature.

Proof. 1. By definition

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = 0 \Longleftrightarrow \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0,\ \text{we have } \frac{u_n}{v_n} < \epsilon. \tag{1.16}$$

Let us choose $\epsilon = 1$, so we have $u_n < v_n$. Since the series $\sum\limits_{n\geq 0} v_n$ converges, the series $\sum\limits_{n\geq 0} u_n$ also converges.
2. Similarly

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = +\infty \Leftrightarrow \lim_{n\to+\infty} \frac{v_n}{u_n} = 0 \Longleftrightarrow \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0,\ \text{we have } \frac{v_n}{u_n} < \epsilon. \tag{1.17}$$

Let us choose $\epsilon = 1$, so we have $u_n > v_n$. Since the series $\sum\limits_{n\geq 0} v_n$ diverges, the series $\sum\limits_{n\geq 0} u_n$ also diverges.
3. If $l \neq 0$ and $\neq +\infty$,

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = l \Longleftrightarrow \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0,\ \text{we have } \left(\frac{u_n}{v_n} - l\right) < \epsilon. \tag{1.18}$$

Let us choose $\epsilon < l$, we therefore have $(l-\epsilon)v_n < u_n < (l+\epsilon)v_n$. Using Corollary 1.2.1 confirms us the reresult, by taking $\alpha = l - \epsilon > 0$ and $\beta = l + \epsilon$. □

1.2.2 Usual rules of convergence

Riemann's rule

Riemann's rule amounts to comparing a series with given positive terms to a Riemann series.

Definition 1.2.2. *A Riemann series is any numerical series whose general term*

$$u_n = \frac{1}{n^\alpha},\ \alpha \in \mathbb{R}$$

.

Proposition 1.2.2. *The Riemann series converges, for all $\alpha > 1$.*

Proof. If $\alpha \geq 0$, $\lim_{n\to+\infty} u_n \neq 0$, the series $\sum\limits_{n\geq1} u_n$ is therefore divergent.
Let us now assume that $\alpha > 0$, and consider the function f, which is defined on $]0, +\infty[$ by $f(x) = \dfrac{1}{x^\alpha}$.
This function is positive defined, continuous and decreasing on $]0, +\infty[$ .By Theorem **??** (see Chapter **??**), the series $\sum\limits_{n\geq1} u_n$ and the generalized integral $\int\limits_1^{+\infty} f(x)dx$ are the same nature. Let

$$F(x) = \int\limits_1^y f(x)dx = \begin{cases} \ln(y), \text{ if } \alpha = 1, \\ \dfrac{1}{(1-\alpha)y^{\alpha-1}} - \dfrac{1}{1-\alpha}, \text{ if } \alpha \neq 1. \end{cases} \tag{1.19}$$

The function F has a finite limit, if and only if $\alpha > 1$, which shows that the series $\sum\limits_{n\geq1} u_n$ converges if and only if $\alpha > 1$. □

Proposition 1.2.3. *(Riemann's rule)*
Let $\sum\limits_{n\geq1} u_n$ be a series with positive real terms and let $\alpha \in \mathbb{R}$. Suppose there exists a positive real number l (or $l = +\infty$) , such that $\lim n^\alpha u_n = l$. We then have:

1. *If $l = 0$ and $\alpha > 1$, the series $\sum\limits_{n\geq1} u_n$ converges.*
2. *If $l = +\infty$ and $\alpha \leq 1$, the series $\sum\limits_{n\geq1} u_n$ diverges.*
3. *If $l \neq 0$ and if $l \neq +\infty$, the two series $\sum\limits_{n\geq1} u_n$ and $\sum\limits_{n\geq1} \dfrac{1}{n^\alpha}$ are the same nature.*

Proof. 1. By definition

$$\lim_{n\to+\infty} n^\alpha u_n = 0 \Longleftrightarrow \forall\epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0,\ \text{we have } n^\alpha u_n < \epsilon. \tag{1.20}$$

Let us choose $\epsilon = 1$, we therefore have $u_n < \dfrac{1}{n^\alpha}$. Since $\sum\limits_{n\geq1} \dfrac{1}{n^\alpha}$ converges for all $\alpha > 1$, the series $\sum\limits_{n\geq1} u_n$ also converges for all $\alpha > 1$.
2. If $l = +\infty$ and $\alpha \leq 1$, still according to the definition of the limit

$$\exists n_1 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_1,\ \text{we have } n^\alpha u_n > \epsilon. \tag{1.21}$$

Let us choose $\epsilon = 1$, we then have $u_n > \frac{1}{n^\alpha}$. Since $\sum_{n\geq 1} \frac{1}{n^\alpha}$ diverges for all $\alpha \leq 1$, the series $\sum_{n\geq 1} u_n$ also diverges for all $\alpha \leq 1$.

3. The third property is the intersection of properties 1 and 2. □

D'Alembert's rule

D'Alembert's rule amounts to comparing a series with positive terms to a geometric series.

Proposition 1.2.4. *Let $\sum_{n\geq 1} u_n$ be a series with strictly positive real terms. Suppose that there exists a positive real number l (or $l = +\infty$), such that $\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = l$. We then have:*

1. *If $l < 1$, the series $\sum_{n\geq 1} u_n$ converges .*
2. *If $l > 1$, the series $\sum_{n\geq 1} u_n$ diverges.*

Proof. By definition

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = l. \iff \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0,\ \text{we have } \left|\frac{u_{n+1}}{u_n} - l\right| < \epsilon. \tag{1.22}$$

1. If $l < 1$, let us choose ϵ, such that $l + \epsilon = k < 1$, we therefore have $\frac{u_{n+1}}{u_n} < k = \frac{v_{n+1}}{v_n}$(by setting $v_n = k^n$). We therefore have:

$$\frac{u_{n+1}}{v_{n+1}} < \frac{u_n}{v_n} < .... < \frac{u_{n_0}}{v_{n_0}} = a\ (a > 0). \tag{1.23}$$

That is, $u_n < av_n$. Since $\sum_{n\geq 1} v_n$ converges, the series $\sum_{n\geq 1} u_n$ converges .
2. If $l > 1$, let us choose ϵ, such that $l - \epsilon = k \geq 1$, we therefore have $\frac{u_{n+1}}{u_n} \geq 1$. Since (u_n) is increasing non-identically zero, we therefore have $\lim_{n\to+\infty} u_n \neq 0$, and then the series $\sum_{n\geq 1} u_n$ diverges. □

Cauchy's rule

Cauchy's rule also amounts to comparing a series with positive terms to a geometric series.

Proposition 1.2.5. *Let* $\sum_{n\geq 1} u_n$ *be a series with strictly positive real terms. Suppose that there exists a positive real number* l *(or* $l = +\infty$*), such that* $\lim_{n\to+\infty} \sqrt[n]{u_n} = l$ *. We then have:*

1. *If* $l < 1$*, the series* $\sum_{n\geq 1} u_n$ *converges .*
2. *If* $l > 1$*, the series* $\sum_{n\geq 1} u_n$ *diverges.*

Proof. By definition

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = l \Longleftrightarrow \forall\epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0,\ \text{we have } \left|\sqrt[n]{u_n} - l\right| < \epsilon. \tag{1.24}$$

1. If $l < 1$, let's choose ϵ, such that $l + \epsilon = k < 1$, we then have $u_n < k^n$. Since $\sum_{n\geq 1} k^n$ converges for all $k < 1$ (geometric series), the series $\sum_{n\geq 1} u_n$ also converges .
2. If $l > 1$, let us choose ϵ, such that $l - \epsilon = k \geq 1$, we therefore have $u_n \geq 1$. Since $\lim_{n\to+\infty} u_n \neq 0$, the series $\sum_{n\geq 1} u_n$ diverges . □

Raabe and Duhamel rule

The Raabe-Duhamel rule amounts to comparing a given series with positive terms to a Riemann series.

Proposition 1.2.6. *Let* $\sum_{n\geq 1} u_n$ *be a series with positive terms. Assume that the following limit exists:*

$$l = \lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right), \tag{1.25}$$

we then have:

1. *If* $l > 1$*, the series* $\sum_{n\geq 1} u_n$ *converges.*
2. *If* $l < 1$*, the series* $\sum_{n\geq 1} u_n$ *diverges.*

Proof. By definition:

$$\lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l \Leftrightarrow \forall\epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0,\ \text{we have: } l - \epsilon < n\left(\frac{u_n}{u_{n+1}} - 1\right) < l + \epsilon. \tag{1.26}$$

1. Suppose that $l > 1$ and choose ϵ, such that $n\left(\frac{u_n}{u_{n+1}} - 1\right) > l - \epsilon = q > 1$.

Let $m \in \mathbb{R}_+^*$ such that $m \in\,]1, q[$, the series of general term $w_n = \frac{1}{n^m}$ is therefore

convergent.
We can write:

$$\frac{w_n}{w_{n+1}} = \left(\frac{n+1}{n}\right)^m = \left(1+\frac{1}{n}\right)^m. \tag{1.27}$$

By performing the limited development of the function $x \mapsto \frac{1}{1+x}$ in the neighborhood of 0, we obtain:

$$\frac{w_n}{w_{n+1}} = 1 + \frac{m}{n} + \frac{1}{n^2}\delta(n), \tag{1.28}$$

where $\frac{\delta(n)}{n} \to 0$, when $n \to +\infty$. That is to say:

$$\text{For all } \eta = q - m > 0,\ \exists n_1 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_1,\ \text{we have: } \frac{\delta(n)}{n} < q - m. \tag{1.29}$$

We then have the following inequalities:

$$m + \frac{\delta(n)}{n} < q < n(\frac{u_n}{u_{n+1}} - 1),\ \text{for all } n \geq \max(n_0, n_1) = n_2, \tag{1.30}$$

or in an equivalent manner:

$$\frac{w_n}{w_{n+1}} = 1 + \frac{m}{n} + \frac{1}{n^2}\delta(n) \leq \frac{u_n}{u_{n+1}}. \tag{1.31}$$

That is to say:

$$\frac{u_{n+1}}{w_{n+1}} \leq \frac{u_n}{w_n} \leq \ldots \leq \frac{u_{n_2}}{w_{n_2}} = a\ (a > 0 \tag{1.32}$$

Since $\sum\limits_{n\geq 1} w_n$ is convergent, it follows from the comparison theorem that $\sum\limits_{n\geq 1} u_n$ is also convergent.

2. Now, suppose that $l < 1$ and choose ϵ, such that:

$$n(\frac{u_n}{u_{n+1}} - 1) < l + \epsilon = q \leq 1,\ \text{for all } n \geq n_0. \tag{1.33}$$

Let us also consider the series with general term $w_n = \frac{1}{n}$, which is divergent. From the inequality (1.33), we can easily see:

$$\frac{u_n}{u_{n+1}} \leq 1 + \frac{1}{n} = \frac{n+1}{n} = \frac{w_n}{w_{n+1}}. \tag{1.34}$$

Since $\sum\limits_{n\geq 1} w_n$ is divergent, it follows from the previous method that $\sum\limits_{n\geq 1} u_n$ is also divergent. □

Bertrand Series

Definition 1.2.3. *A Bertrand series is any numerical series whose general term*

$$u_n = \frac{\ln^{\beta}(n)}{n^{\alpha}}, \ \alpha \in \mathbb{R}_*^+ \text{ and } \beta \in \mathbb{R}.$$

Proposition 1.2.7. *The Bertrand series is:*

1. *convergent, if and only if*

   1.1. $\alpha > 1$ *and* $\beta \in \mathbb{R}$,

   or else

   1.2. $\alpha = 1$ *and* $\beta < -1$.

2. *divergent if and only if*

   2.1. $\alpha < 1$ *and* $\beta \in \mathbb{R}$,

   or else

   2.2. $\alpha = 1$ *and* $\beta \geq -1$.

Proof. 1.1. Let's assume that $\alpha > 1$ and $\beta \in \mathbb{R}$. So it exists $\gamma \in]1, \alpha[$ verifying

$$\lim_{n\to+\infty} n^{\gamma} u_n = \lim_{n\to+\infty} \frac{\ln^{\beta}(n)}{n^{\alpha-\gamma}} = 0, \ (\text{since } \alpha - \gamma > 0). \tag{1.35}$$

Since $\gamma > 1$, the use of Proposition 1.2.3 confirms the convergence of the considered series.

1.2 and 2.2. Now suppose that $\alpha = 1$. We can write $u_n = \dfrac{\ln^{\beta}(n)}{n}$.

* If $\beta = 0$, $\sum\limits_{n\geq 1} u_n = \sum\limits_{n\geq 1} \dfrac{1}{n}$ which is divergent.

* If $\beta > 0$, we have:

$$\lim_{n\to+\infty} n u_n = \lim_{n\to+\infty} \ln^{\beta}(n) = +\infty, \tag{1.36}$$

and the series $\sum\limits_{n\geq 1} u_n$ is divergent by applying comparaison theorem.

* If $\beta < 0$, consider the function f defined by:

$$f(x) = \frac{\ln^{\beta}(x)}{x^{\alpha}}, x \in]\delta, +\infty[\ (\delta > 1). \tag{1.37}$$

This function is positive defined, continuous and decreasing on $]\delta, +\infty[$. According to Cauchy's theorem, the series $\sum\limits_{n\geq 2} \dfrac{\ln^{\beta}(n)}{n^{\alpha}}$ and the generalized

integral $\int_{\delta}^{+\infty} \frac{\ln^{\beta}(x)}{x^{\alpha}} dx$ are of the same nature.
We know that

$$\int_{\delta}^{t} \frac{\ln^{\beta}(x)}{x^{\alpha}} dx = \begin{cases} \frac{1}{\beta+1}\left[\ln^{\beta+1}(t) - \ln^{\beta+1}(\delta)\right], \text{ if } \beta \neq -1, \\ \ln\left(\frac{\ln t}{\ln \delta}\right), \text{ if } \beta = -1. \end{cases} \tag{1.38}$$

Which gives:

$$\lim_{t \to +\infty} \int_{\delta}^{t} \frac{\ln^{\beta}(x)}{x^{\alpha}} dx = \begin{cases} -\frac{\ln^{\beta+1}(\delta)}{\beta+1}, \text{ if } \beta < -1, \\ +\infty, \text{ if } \beta > -1, \\ +\infty, \text{ if } \beta = -1 \end{cases} \tag{1.39}$$

So, if$\alpha = 1$ and $\beta < -1$, the integral $\int_{\delta}^{+\infty} \frac{\ln^{\beta}(x)}{x^{\alpha}} dx$ converges, whereas if $\alpha = 1$ and $\beta \geq -1$, the integral $\int_{\delta}^{+\infty} \frac{\ln^{\beta}(x)}{x^{\alpha}} dx$ diverges.
2.2 Now, let us suppose that $\alpha < 1$ and $\beta \in \mathbb{R}$. So it exists $\gamma \in]\alpha, 1[$ verifying

$$\lim_{n \to +\infty} n^{\gamma} u_n = \lim_{n \to +\infty} n^{\gamma - \alpha} \ln^{\beta}(n) = +\infty, \text{ since } \gamma - \alpha > 0. \tag{1.40}$$

Since $\gamma < 1$, the use of Proposition 1.2.3 confirms the divergence of the series $\sum_{n \geq 1} u_n$. □

1.3 Series of arbitrary sign

1.3.1 Convergence rules for series of arbitrary sign

Abel's Rule

Proposition 1.3.1. *Let (b_n) be a positive sequence decreasing towards 0, and let (a_n) be a sequence verifying:*

$$\exists M > 0, \forall n \in \mathbb{N}, \left| A_n = \sum_{k=1}^{n} a_k \right| \leq M. \tag{1.41}$$

Then, the series of general term $u_n = a_n b_n$ is convergent.

Proof. We can write:

$$\begin{aligned}\sum_{n\geq 1}^{+\infty} u_n &= a_1(b_1-b_2)+(a_1+a_2)(b_2-b_3)+\ldots+(a_1+a_2+\ldots+a_n)(b_n-b_{n-1})+\ldots\\ &= \sum_{n\geq 1}^{+\infty}(a_1+a_2+\ldots+a_n)(b_n-b_{n+1})\\ &= \sum_{n\geq 1}^{+\infty} A_n(b_n-b_{n+1}). \qquad (1.42)\end{aligned}$$

We can then see the series of general terms appear:

$$v_n = A_n(b_n-b_{n+1}). \qquad (1.43)$$

We just need to show that this series is convergent. Indeed:

$$\begin{aligned}|v_n| &= |A_n(b_n-b_{n+1})|\\ &\leq M(b_n-b_{n+1}),\ \text{car } b_n-b_{n+1}\geq 0\ ((b_n) \text{ is decreasing}). \qquad (1.44)\end{aligned}$$

The series $\sum\limits_{n\geq 1}^{+\infty}(b_n-b_{n+1})$ is convergent, since the sequence of its partial sums is verified:

$$\sum_{k=1}^{n}(b_k-b_{k+1}) = b_1-b_{n+1}\to b_1,\ \text{as } n\to+\infty. \qquad (1.45)$$

The comparison theorem asserts the absolute convergence of the series $\sum\limits_{n\geq 1}^{+\infty} v_n$. It follows that the starting series $\sum\limits_{n\geq 1}^{+\infty} u_n$ is convergent. □

1.3.2 Alternating series

Definition 1.3.1. *An alternating series is any series whose general term*

$$u_n = (-1)^n a_n, a_n \geq 0,\ \textit{for all } n\in\mathbb{N}. \qquad (1.46)$$

Convergence rule for alternating series

Proposition 1.3.2. *(Leibniz criterion)*

Let $\sum\limits_{n\geq 0} u_n$ be an alternating series. If $(|u_n|$ is a sequence decreasing towards 0, then *the series* $\sum_{n\geq 0} u_n$ *is convergent, moreover we have:*

$$\left|\sum_{k=n+1}^{+\infty} (-1)^k u_k\right| \leq u_{n+1}. \tag{1.47}$$

Proof. The series $\sum_{n\geq 0} u_n$ is alternating, so we can write:

$$u_n = u_n = a_n \times b_n, \text{ such thate } b_n = (-1)^n \text{ and } a_n \geq 0, \text{ for all } n \in \mathbb{N}. \tag{1.48}$$

Since $\left|\sum_{k=0}^{n} b_k\right| \leq 1$ and ($|u_n| = a_n$) a sequence decreasing towards 0, the use of Abel's criterion shows the convergence of the series $\sum_{n\geq 0} u_n$.
Let now (S_n) be the partial sum of the series $\sum_{n\geq 0} u_n$, we then have

$$S_{2n+2} - S_{2n} = u_{2n+2} - u_{2n+1} \leq 0, \tag{1.49}$$

$$S_{2n+1} - S_{2n-1} = -u_{2n+1} + u_{2n} \geq 0, \text{ for all } n \in \mathbb{N}. \tag{1.50}$$

So the sequence (S_{2n}) is decreasing and the sequence (S_{2n+1}) is increasing. Moreover we have:

$$S_{2n+1} - S_{2n} = -u_{2n+1}, \text{ for all } n \in \mathbb{N}. \tag{1.51}$$

Which shows that the sequence $(S_{2n+1} - S_{2n})$ tends to 0. The sequences (S_{2n}) and (S_{2n+1}) are therefore adjacent and thus both converge to the same finite limit S. Consequently the sequence (S_n) converges to S, which shows once again that the series $\sum_{n\geq 0} u_n$ converges. Furthermore, we have:

$$S_{2n+1} \leq S \leq S_{2n+2} = S_{2n+1} + u_{2n+2}, \text{ for all } n \in \mathbb{N}. \tag{1.52}$$

That is to say

$$S - S_{2n+2} =\leq u_{2n+2}, \text{ for all } n \in \tag{1.53}$$

We also have

$$S_{2n} - u_{2n+1} = S_{2n+1} \leq S \leq S_{2n}, \text{ for all } n \in \mathbb{N}, \tag{1.54}$$

or in an equivalent manner:

$$-\,u_{2n+1} \leq S - S_{2n} \leq 0, \text{ for all } n \in \mathbb{N}. \tag{1.55}$$

It follows that, for all $n \in \mathbb{N}$:

$$\left|\sum_{k=n+1}^{+\infty}(-1)^k u_k\right| = |S - S_n| \leq u_{n+1}, \tag{1.56}$$

where S is the sum of the series. □

1.3.3 Absolutely convergent series

Definition 1.3.2. *A series with general term u_n is said to be absolutely convergent if the series with general term $|u_n|$ converges.*

Proposition 1.3.3. *If the numerical series with general term u_n converges absolutely, then this series is convergent.*

Proof. Suppose that the series of general term u_n converges absolutely. Applying the Cauchy criterion to the series of general term $|u_n|$, we find:

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall p, q \in \mathbb{N},\ p > q \geq n_0,\ \text{we have } \sum_{k=q+1}^{p} |u_k| < \epsilon. \tag{1.57}$$

On the other hand, we have:

$$\left|\sum_{k=q+1}^{p} u_k\right| \leq \sum_{k=q+1}^{p} |u_k| < \epsilon. \tag{1.58}$$

The series of the general term u_n then verifies the Cauchy criterion. This series is therefore indeed convergent. □

Remak 1.4. *The converse of this proposition is false. For example, the series with general term $\frac{(-1)^n}{n}$ converges, while it does not converge absolutely.*

Proposition 1.3.4. *Let $\sum_{n\geq 0} u_n$ be a numerical series with arbitrary terms. Suppose that there exists a positive numerical sequence verifying:*

$$\exists n_0 \in \mathbb{N}, \forall n \geq n_0,\ \text{we have } |u_n| \leq v_n, \tag{1.59}$$

then, if $\sum_{n\geq 0} v_n$ converges, $\sum_{n\geq 0} u_n$ converges absolutely.

Proof. The proof proceeds immediately, using the comparison theorem of series with positive terms. □

1.3.4 Semi-convergent series

Definition 1.3.3. *A series with general term u_n is said to be semi-convergent, when it converges without being absolutely convergent.*

We also have the following immediate properties:

Proposition 1.3.5. *Suppose that the two series of respective general terms u_n and v_n are absolutely convergent (respectively semi-convergent), then the sum series of general term $(u_n + v_n)$ is absolutely convergent (respectively semi-convergent), and the series produced by a scalar of general term αu_n is absolutely convergent (respectively semi-convergent), for all $\alpha \in \mathbb{k}(\mathbb{k} = \mathbb{R}$ or $\mathbb{C})$.*

Proof. The proof of this proposition proceeds immediately using the Cauchy criterion. □

Example 1.3.1. *It can be easily shown that the alternating Riemann series $\sum_{n\geq 1} \frac{(-1)^n}{n^\alpha}$, $\alpha \in \mathbb{R}$ is:*

* *divergent for all $\alpha \leq 0$,*

* *absolutely convergent, for all $\alpha > 1$,*

* *semi-convergent, for all $0 < \alpha \leq 1$.*

1.3.5 Additional properties of series convergent

Property 1: Use of the D'Alembert and Cauchy criteria

We know that these two criteria apply a priori to series with positive terms. Let $\sum_{n\geq 0} u_n$ be a series with of arbitray sign. We can therefore perfectly use these criteria with any series of terms, but we must be very careful not to forget the absolute values.
Let us now suppose that $l_1 = \lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right|$ exists (respectively $l_2 = \lim_{n\to+\infty} \sqrt[n]{|u_n|}$ exists).

* If $l_1 < 1$ (respect. $l_2 < 1$), the D'Alembert criterion (resp. the Cauchy criterion) asserts that the series $\sum_{n\geq 0} |u_n|$ is convergent. The $\sum_{n\geq 0} u_n$ is then absolutely convergent.

* If $l_1 > 1$ (respect. $l_2 > 1$), the D'Alembert criterion (resp. the Cauchy criterion) asserts that the general term $|u_n|$ tends to $+\infty$. The series $\sum\limits_{n\geq 0} u_n$ is then divergent.

Example 1.3.2. *Consider the series of general term* $u_n = \left(\frac{3n+1}{n+1}\right)^{3n} \alpha^n$, $\alpha \in \mathbb{R}$ *and* $n \geq 0$.
We have $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = 27\,|\alpha|$. *the Cauchy criterion states that:*

1. *If* $|\alpha| < \frac{1}{27}$, *the considered series is absolutely convergent.*
2. *1. If* $|\alpha| > \frac{1}{27}$, *the considered series is divergent.*
3. *If 1. If* $|\alpha| = \frac{1}{27}$, *the value absolute of general term becomes*

$$|u_n| = \left(\frac{3n+1}{3n+3}\right)^{3n} = \left(1 - \frac{2}{3n+3}\right)^{3n} \to \exp(-2)(\neq 0), \textit{ when } n \to +\infty,$$

and the considered series is divergent.

Property 3: Use of Limited Developments

By performing a limited development of the general term u_n of the series $\sum\limits_{n\geq 0} u_n$ in the neighborhood of infinity at a sufficiently high order (to have an absolutely convergent remainder), we can quickly conclude on the nature of this series.
As an example, the series of general term $u_n = \frac{(-1)^n}{n+(-1)^n}$. We can éwrite:

$$u_n = \frac{(-1)^n}{n} \times \frac{1}{1+\frac{(-1)^n}{n}}. \tag{1.60}$$

By performing the limited development of the function $x \mapsto \frac{1}{1+x}$ in the neighborhood of 0, we obtain:

$$\begin{aligned} u_n &= \frac{(-1)^n}{n}\left(1 - \frac{(-1)^n}{n} + \frac{1}{n}\epsilon(n)\right) \\ &= \frac{(-1)^n}{n} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\epsilon(n), \text{ où } \epsilon(n) \to 0, \text{ when } n \to +\infty. \end{aligned} \tag{1.61}$$

We therefore have the sum of two convergent series and an absolutely convergent remainder. The considered series is therefore convergent.

1.4 Cauchy product of series

Definition 1.4.1. *We consider two numerical series with general terms u_n and v_n, respectively. We call the Cauchy product of these two series the series with general term:*

$$w_n = \sum_{k=0}^{n} u_k v_{n-k}. \tag{1.62}$$

Proposition 1.4.1. *Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} u_n$ be two absolutely convergent series. Then the Cauchy product series with general term defined by (1.62) is absolutely convergent, and moreover, we have:*

$$\sum_{n\geq 0} w_n = \left(\sum_{n\geq 0} u_n\right) \times \left(\sum_{n\geq 0} u_n\right). \tag{1.63}$$

Proof. For all $n \in \mathbb{N}$, Let us consider the following partial sums:

$$S_n = \sum_{k=0}^{n} u_k, T_n = \sum_{k=0}^{n} v_n \text{ and } R_n = \sum_{k=0}^{n} w_n. \tag{1.64}$$

Let us also consider the following notations:

$$S = \sum_{n\geq 0} |u_n| \text{ and } T = \sum_{n\geq 0} |v_n|. \tag{1.65}$$

The two sequences (S_n) and (T_n) are convergent, so they satisfy the following Cauchy criterion:

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall p, n \in \mathbb{N},\ p > n \geq n_0,\ \text{we have } \left|\sum_{k=n+1}^{p} u_k\right| < \epsilon \text{ and } \left|\sum_{k=n+1}^{p} v_k\right| < \epsilon, \tag{1.66}$$

On the one hand, $R_{2n} - S_n T_n$ can be rewritten in the form:

$$\begin{aligned} R_{2n} - S_n T_n &= u_0(v_{n+1} + ... + v_{2n}) + u_1(v_{n+1} + ... + v_{2n-1}) + ... + u_{n-1}v_{n+1} \\ &\quad + v_0(u_{n+1} + ... + u_{2n}) + v_1(u_{n+1} + ... + u_{2n-1}) + ... + v_{n-1}u_{n+1}. \end{aligned}$$

For all $p, n \in \mathbb{N}$, $p > n \geq n_0$, we then have

$$\begin{aligned} |R_{2n} - S_n T_n| &\leq \epsilon \left(\sum_{k=0}^{n} |u_k| \ \sum_{k=0}^{n} |v_k|\right) \\ &\leq \epsilon(S + T). \end{aligned} \tag{1.67}$$

So (R_{2n}) is convergent, and furthermore, we have:

$$\lim_{n\to+\infty} R_{2n} = \lim_{n\to+\infty} S_n T_n = \left(\sum_{n\geq 0} u_n\right) \times \left(\sum_{n\geq 0} u_n\right). \tag{1.68}$$

On the other hand, for all $n \geq n_0$, we have:

$$\begin{aligned} |R_{2n+1} - R_{2n}| &= |(u_0 v_{2n+1} + \ldots + u_{n+1} v_n) + (v_0 u_{2n+1} + \ldots + v_{n+1} u_n)| \\ &\leq \epsilon(S + T), \end{aligned} \tag{1.69}$$

which ensures the convergence of (R_{2n+1}).

Since (R_{2n}) and (R_{2n+1}) are convergent, (R_n) is also convergent, and moreover:

$$\lim_{n\to+\infty} R_n = \lim_{n\to+\infty} R_{2n+1} = \lim_{n\to+\infty} R_{2n} = \left(\sum_{n\geq 0} u_n\right) \times \left(\sum_{n\geq 0} u_n\right). \tag{1.70}$$

That is to say:

$$\left(\sum_{n\geq 0} w_n\right) = \left(\sum_{n\geq 0} u_n\right) \times \left(\sum_{n\geq 0} u_n\right).$$

□

1.5 Exercises about chapter 1

Exercise 1.5.1. *Show that the following numerical series are convergent and calculate their sums:*

$$1) \sum_{n=2}^{+\infty} \frac{1}{n(n-1)} \qquad 2) \sum_{n=0}^{+\infty} \frac{n^2}{n!} \qquad 3) \sum_{n=0}^{+\infty} (-1)^{n+1} \frac{\cos(nx)}{2^n},\ x \in \mathbb{R}.$$

Exercise 1.5.2. *Study the nature of the following numerical series:*

$$1) \sum_{n=1}^{+\infty} n \sin\left(\frac{1}{n}\right) \qquad 2) \sum_{n=1}^{+\infty} \arctan\left(\frac{1}{n^2}\right) \qquad 3) \sum_{n=1}^{+\infty} \frac{\alpha^n}{\alpha^{2n} + \alpha^n + 1}\ (\alpha \geq 0)$$

$$4) \sum_{n=1}^{+\infty} \left(\frac{n+a}{n+b}\right)^{n^2},\ a \text{ et } b \in \mathbb{R} \qquad 5) \sum_{n=1}^{+\infty} \frac{\ln(n)}{n^2+2} \qquad 6) \sum_{n=1}^{+\infty} \frac{1 \times 3 \times \ldots \times (2n-1)}{2 \times 4 \times \ldots \times (2n)}$$

$$7) \sum_{n\geq 1} \frac{2^n}{n^2} \sin^{2n}(\theta),\ \theta \in \left[0, \frac{\pi}{2}\right] \qquad 8) \frac{\exp(inx)}{n},\ x \in \mathbb{R}.$$

Exercise 1.5.3. *Let θ be a real number. Study, according to the value of θ, the absolute convergence, the semi-convergence and the divergence of the following numerical series:*

$$1)\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^\theta},\qquad 2)\sum_{n=1}^{+\infty}\left(\frac{2n-1}{n+1}\right)^{2n}\theta^n\,,\qquad 3)\sum_{n=1}^{+\infty}\frac{\cos(n)}{n^\theta+\cos(n)}.$$

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