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Algebra II, Worksheet 4 answers

Exercise No. 1 :

Matrix Multiplication : Let $A = \left(a_{ij}\right) \in \mathcal{M}_{m,n}(\mathbb{K})$ and $B = \left(b_{ij}\right) \in \mathcal{M}_{n,p}(\mathbb{K})$. The product (or multiplication) of A and B, denoted $A \cdot B$, is the matrix $C = \left(c_{ij}\right)_{\substack{1\le i\le m\\ 1\le j\le p}} \in \mathcal{M}_{m,p}(\mathbb{K})$, where each entry c_{ij} is defined by

$$\forall (i,j) \in \{1,2,...,m\} \times \{1,2,...,p\} : c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}.$$

In other words, the entry c_{ij} of the product $A\cdot B$ is obtained by summing the products of the entries from the $i-th$ row of A with the corresponding entries from the $j-th$ column of B. This can also be interpreted as the dot product of the $i-th$ row of A and the $j-th$ column of B.

Matrix multiplication is defined if and only if the number of columns of the first matrix is equal to the number of rows of the second matrix. This condition is given by the following rule

$$\text{Matrix of size } (m\times n)\cdot \text{ Matrix of size } (n\times p) = \text{ Matrix of size } (m\times p).$$

1. (a) Computing A^2 and A^3 (Powers of a Square Matrix or Matrix Powers).
We have

$$A^2 = A\cdot A = \begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\cdot\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix} = \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}.$$

and

$$A^3 = A^2\cdot A = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} = O_3$$

where O_3 is the zero matrix of order 3 . Hence, for all integers $n \geq 3$, we have

$$A^n = A^{n-3}\cdot A^3 = A^{n-3}\cdot O_3 = O_3 \tag{1}$$

2. **(a) Explicit Form of** $M(x)$. For all $x\in\mathbb{R}$, we have

$$M(x) = I_3 + xA + \frac{x^2}{2}A^2$$

$$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + x\cdot\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix} + \frac{x^2}{2}\cdot\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}1&x&\frac{1}{2}x^2\\0&1&x\\0&0&1\end{pmatrix}$$

Thus, the explicit form of $M(x)$ is

$$M(x) = \begin{pmatrix}1&x&\frac{1}{2}x^2\\0&1&x\\0&0&1\end{pmatrix}.$$

(b) (j) Let $x, y\in\mathbb{R}$, then

$$M(x)\cdot M(y) = \left(I_3 + xA + \frac{x^2}{2}A^2\right)\cdot\left(I_3 + yA + \frac{y^2}{2}A^2\right)$$

$$= I_3^2 + yA + \frac{y^2}{2}A^2 + xA + xyA^2 + \frac{xy^2}{2}A^3 + \frac{x^2}{2}A^2 + \frac{x^2y}{2}A^3 + \frac{x^2y^2}{4}A^4.$$

Using the identity 1 , we obtain

$$M(x)\cdot M(y) = I_3 + yA + \frac{y^2}{2}A^2 + xA + xyA^2 + \frac{x^2}{2}A^2 = I_3 + (x+y)A + \left(xy + \frac{x^2}{2} + \frac{y^2}{2}\right)A^2$$

$$= I_3 + (x+y)A + \frac{1}{2}(x+y)^2A^2$$

$$= M(x+y).$$

Alternative Method : We have

$$M(x).M(y) = \begin{pmatrix} 1 & x & \frac{1}{2}x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.\begin{pmatrix} 1 & y & \frac{1}{2}y^2 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y & xy+\frac{1}{2}x^2+\frac{1}{2}y^2 \\ 0 & 1 & x+y \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x+y & \frac{1}{2}(x+y)^2 \\ 0 & 1 & x+y \\ 0 & 0 & 1 \end{pmatrix} = M(x+y).$$

(jj) Let $x \in \mathbb{R}$, then

$$M(x).M(x') = I_3 \Longrightarrow M(x+x') = I_3$$

where

$$I_3 = M(0).$$

Therefore

$$M(x+x') = M(0)$$

$$\Longrightarrow x + x' = 0$$

$$\Longrightarrow x' = -x \in \mathbb{R}.$$

(jjj) **Inverse of M(x).** Let $x \in \mathbb{R}$. The matrix $M(x)$ is invertible if and only if there exists a matrix $B \in \mathcal{M}_3(\mathbb{R})$ such that

$$M(x)\cdot B = B\cdot M(x) = I_3.$$

Moreover, if $M(x)$ is invertible, then B is unique and $B = M^{-1}(x)$. We have

$$M(x)\cdot M(-x) = M(x-x) = M(0) = I_3$$

Therefore, the matrix $M(x)$ is invertible and its inverse is

$$M^{-1}(x) = M(-x) = \begin{pmatrix} 1 & -x & \frac{1}{2}x^2 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise No. 2 :

1. Recall that the transpose of a matrix satisfies the following properties

$$\forall \lambda \in \mathbb{R}, \forall A, B \in \mathcal{M}_n(\mathbb{R}) : \begin{cases} {}^t(A+B) = {}^tA + {}^tB \\ {}^t(\lambda\cdot A) = \lambda\cdot {}^tA \end{cases}$$

In other words, the transpose operation is linear.

If $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$, its transpose is defined as

$${}^tA = (a_{ji}) \in \mathcal{M}_n(\mathbb{R})$$

That is, each row becomes a column, and each column becomes a row in the transposed matrix.

(a) For F_1., the set of symmetric matrices.
(j) The zero matrix O_n satisfies ${}^tO_n = O_n$, so $O_n \in F_1$.
(jj) Closed Under Linear Combinations : Let $\alpha, \beta \in \mathbb{R}$ and $A, B \in F_1$,then

$$ {}^tA = A, {}^t B = B. $$

We obtain

$$ {}^t(\alpha A + \beta B) = \alpha {}^tA + \beta {}^tB = \alpha A + \beta B \in F_1. $$

So F_1 is a vector subspace of $\mathcal{M}_n(\mathbb{R})$.
In the same way, we can show that F_2, the set of antisymmetric matrices or Skew-Symmetric matrices, is also a vector subspace of $\mathcal{M}_n(\mathbb{R})$.
2. We have

$$ \begin{cases} {}^tB_1 ={}^t \left(\frac{1}{2}({}^tA + A)\right) = \frac{1}{2}\left({}^t({}^tA) +{}^t A\right) = \frac{1}{2}({}^tA + A) = B_1 \\ {}^tB_2 ={}^t \left(\frac{1}{2}(A -{}^t A)\right) = \frac{1}{2}\left({}^tA -{}^t ({}^tA)\right) = \frac{1}{2}({}^tA - A) = -B_2 \end{cases} $$

$$ \Longrightarrow \begin{cases} B_1 \in F_1 \\ B_2 \in F_2 \end{cases}. $$

3. Decomposition of a Matrix into Symmetric and Antisymmetric Parts
We observe that any matrix $A \in \mathcal{M}_n(\mathbb{R})$ can be written as

$$ A = B_1 + B_2 = \frac{1}{2}({}^tA + A) + \frac{1}{2}(A -{}^t A), $$

with $B_1 \in F_1$ and B_2. Then the decomposition of A is

$$ A = B_1 + B_2 $$

with

$$ B_1 = \frac{1}{2}({}^tA + A) = \frac{1}{2}(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}) = \frac{1}{2}\begin{pmatrix} 2 & 5 \\ 5 & 8 \end{pmatrix} $$

and

$$ B_2 = \frac{1}{2}(A -{}^t A) = \frac{1}{2}(\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

Exercise No. 3 : Exercise No. 3 : Consider the linear mapping f defined by

$$ \begin{array}{llll} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (x, y, z) & \longmapsto & f(x, y, z) = (3x - y + z, -x - 2y - 5z, x + y + 3z) \end{array} $$

Let $B = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ be the canonical (or standard) basis of $\mathbb{R}^3$.
1. Find the matrix $A = \underset{B}{Mat}(f)$ associated with f with respect to the basis B.
2. Let $B' = \left\{e'_1 = (-4, 1, 3), e'_2 = (2, 0, -1), e'_3 = (-1, 1, 1)\right\}$ be a new basis of $\mathbb{R}^3$.
(a) Determine the change-of-basis matrix P from B to B' and compute its inverse P^{-1}.
(b) For the vector $v = (1, 2, -1) \in \mathbb{R}^3$, find the coordinates of v in the new basis B'.
(c) Compute the matrix $A' = \underset{B'}{Mat}(f)$ associated with f with respect to the basis B'.

Solution :
1. The matrix associated with the linear mapping f, denoted by $A = \underset{B}{Mat}(f)$, is the matrix of size $m \times n$ where $(m = \dim(F = \mathbb{R}^3 = 3), n = \dim(E = \mathbb{R}^3) = 3)$ and whose columns are the coordinates of the vectors $f(e_1), f(e_2)$ and $f(e_3)$ expressed in the basis B. That is

$$ A = \underset{B}{Mat}(f) = =\underset{B}{Mat}\begin{pmatrix} f(e_1) & f(e_2) & f(e_3) \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}). $$

where $\underset{B}{Mat}\begin{pmatrix} f(e_1) & f(e_2) & f(e_3) \end{pmatrix}$ is the matrix associated with the family of vectors $\{f(e_1), f(e_2), f(e_3)\}$.

We compute the images of the basis vectors

$$\begin{aligned} f(e_1) &= f(1,0,0) = (3,-1,1) = 3e_1 - e_2 + e_3 \\ f(e_2) &= f(0,1,0) = (-1,-2,1) = -e_1 - 2e_2 + e_3 \\ f(e_3) &= f(0,0,1) = (1,-5,3) = e_1 - 5e_2 + 3e_3 \end{aligned}$$

From this, we conclude that the matrix of f in the canonical basis B is

$$A = \underset{B}{Mat}(f) = \begin{pmatrix} 3 & -1 & 1 \\ -1 & -2 & -5 \\ 1 & 1 & 3 \end{pmatrix}.$$

2. (a) The change-of-basis matrix P from an old basis B to a new basis B' (also called the transition matrix), denoted $\underset{B\longrightarrow B'}{P}$, is the square matrix whose columns are the coordinates of the vectors of the new basis expressed in the old basis B. That is

$$P = \underset{B\longrightarrow B'}{P} = \underset{B}{Mat}(e_1', e_2', e_3') \in \mathcal{M}_3(\mathbb{R}) \text{ (is the } 3\times 3 \text{ square matrix)},$$

where $\underset{B}{Mat}(\cdot)$ denotes the matrix of a family of vectors with respect to the basis B. We get

$$P = \underset{B\longrightarrow B'}{P} = \underset{B}{Mat}(e_1', e_2', e_3') = \begin{pmatrix} -4 & 2 & -1 \\ 1 & 0 & 1 \\ 3 & -1 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}).$$

Compute P^{-1}. The matrix P^{-1} is the change-of-basis matrix from the new basis B' to the old basis B. In other words

$$P^{-1} = \underset{B'\longrightarrow B}{P} = \underset{B'}{Mat}(e_1, e_2, e_3),$$

which means that the columns of P^{-1} are the coordinates of the vectors e_1, e_2, e_3 in the basis B'. Therefore, for each vector e_i in the basis B, we seek scalars $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$e_i = \lambda_1 e_1' + \lambda_1 e_2' + \lambda_3 e_3'.$$

Finding the coordinates of e_1 in the basis B'. We solve for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$e_1 = \lambda_1 e_1' + \lambda_1 e_2' + \lambda_3 e_3'.$$

This leads to the system of equations

$$\begin{cases} -4\lambda_1 + 2\lambda_2 - \lambda_3 = 1 \\ \lambda_1 + \lambda_3 = 0 \\ 3\lambda_1 - \lambda_2 + \lambda_3 = 0 \end{cases} \Longrightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1$$

Finding the coordinates of e_2 in the basis B'. We solve for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$e_2 = \lambda_1 e_1' + \lambda_1 e_2' + \lambda_3 e_3'.$$

This leads to the system of equations

$$\begin{cases} -4\lambda_1 + 2\lambda_2 - \lambda_3 = 0 \\ \lambda_1 + \lambda_3 = 1 \\ 3\lambda_1 - \lambda_2 + \lambda_3 = 0 \end{cases} \Longrightarrow \lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 2.$$

Finding the coordinates of e_3 in the basis B'. We solve for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$e_3 = \lambda_1 e_1' + \lambda_1 e_2' + \lambda_3 e_3'.$$

This leads to the system of equations

$$\begin{cases} -4\lambda_1 + 2\lambda_2 - \lambda_3 = 0 \\ \lambda_1 + \lambda_3 = 0 \\ 3\lambda_1 - \lambda_2 + \lambda_3 = 1 \end{cases} \Longrightarrow \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = -2$$

Thus, we obtain

$$P^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 3 \\ -1 & 2 & -2 \end{pmatrix}.$$

(b) Finding the coordinates of v in the basis B'. Let $X_B = \underset{B}{Mat}(v) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ be the column matrix of the coordinates of v in the basis B, and let $X'_{B'} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$ be the column matrix of the coordinates of v in the basis B'. The change-of-basis formula gives

$$X_B = P \cdot X'_{B'} \Longleftrightarrow X'_{B'} = P^{-1} \cdot X_B \Longleftrightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 3 \\ -1 & 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 5 \end{pmatrix}$$

Thus, the coordinates of $v = (1, 2, -1)$ in the basis B' are $(-3, -3, 5)$.

(c) The matrix associated with f relative to the new basis B' is given by

$$A' = \underset{B'}{Mat}(f) = P^{-1} \cdot A \cdot P$$

$$= \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 3 \\ -1 & 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 & 1 \\ -1 & -2 & -5 \\ 1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} -4 & 2 & -1 \\ 1 & 0 & 1 \\ 3 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & 0 & 9 \\ 11 & 4 & 9 \\ -28 & 3 & -15 \end{pmatrix}$$