# 1

# Linear Systems

This chapter presents a study of linear systems of ordinary differential equations:

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

where $\mathbf{x} \in \mathbf{R}^n$, $A$ is an $n \times n$ matrix and

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}.$$

It is shown that the solution of the linear system (1) together with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

where $e^{At}$ is an $n \times n$ matrix function defined by its Taylor series. A good portion of this chapter is concerned with the computation of the matrix $e^{At}$ in terms of the eigenvalues and eigenvectors of the square matrix $A$. Throughout this book all vectors will be written as column vectors and $A^T$ will denote the transpose of the matrix $A$.

## 1.1 Uncoupled Linear Systems

The method of separation of variables can be used to solve the first-order linear differential equation

$$\dot{x} = ax.$$

The general solution is given by

$$x(t) = ce^{at}$$

where the constant $c = x(0)$, the value of the function $x(t)$ at time $t = 0$.

Now consider the uncoupled linear system

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= 2x_2. \end{aligned}$$

This system can be written in matrix form as

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note that in this case $A$ is a diagonal matrix, $A = \operatorname{diag}[-1, 2]$, and in general whenever $A$ is a diagonal matrix, the system (1) reduces to an uncoupled linear system. The general solution of the above uncoupled linear system can once again be found by the method of separation of variables. It is given by

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_2 e^{2t} \end{aligned} \tag{2}$$

or equivalently by

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \mathbf{c} \tag{2$'$}$$

where $\mathbf{c} = \mathbf{x}(0)$. Note that the solution curves (2) lie on the algebraic curves $y = k/x^2$ where the constant $k = c_1^2 c_2$. The solution (2) or (2′) defines a motion along these curves; i.e., each point $\mathbf{c} \in \mathbf{R}^2$ moves to the point $\mathbf{x}(t) \in \mathbf{R}^2$ given by (2′) after time $t$. This motion can be described geometrically by drawing the solution curves (2) in the $x_1, x_2$ plane, referred to as the *phase plane,* and by using arrows to indicate the direction of the motion along these curves with increasing time $t$; cf. Figure 1. For $c_1 = c_2 = 0$, $x_1(t) = 0$ and $x_2(t) = 0$ for all $t \in \mathbf{R}$ and the origin is referred to as an *equilibrium point* in this example. Note that solutions starting on the $x_1$-axis approach the origin as $t \to \infty$ and that solutions starting on the $x_2$-axis approach the origin as $t \to -\infty$.

The *phase portrait* of a system of differential equations such as (1) with $\mathbf{x} \in \mathbf{R}^n$ is the set of all solution curves of (1) in the phase space $\mathbf{R}^n$. Figure 1 gives a geometrical representation of the phase portrait of the uncoupled linear system considered above. The *dynamical system* defined by the linear system (1) in this example is simply the mapping $\phi\colon \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$ defined by the solution $\mathbf{x}(t, \mathbf{c})$ given by (2′); i.e., the dynamical system for this example is given by

$$\phi(t, \mathbf{c}) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \mathbf{c}.$$

Geometrically, the dynamical system describes the motion of the points in phase space along the solution curves defined by the system of differential equations.

The function

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x}$$

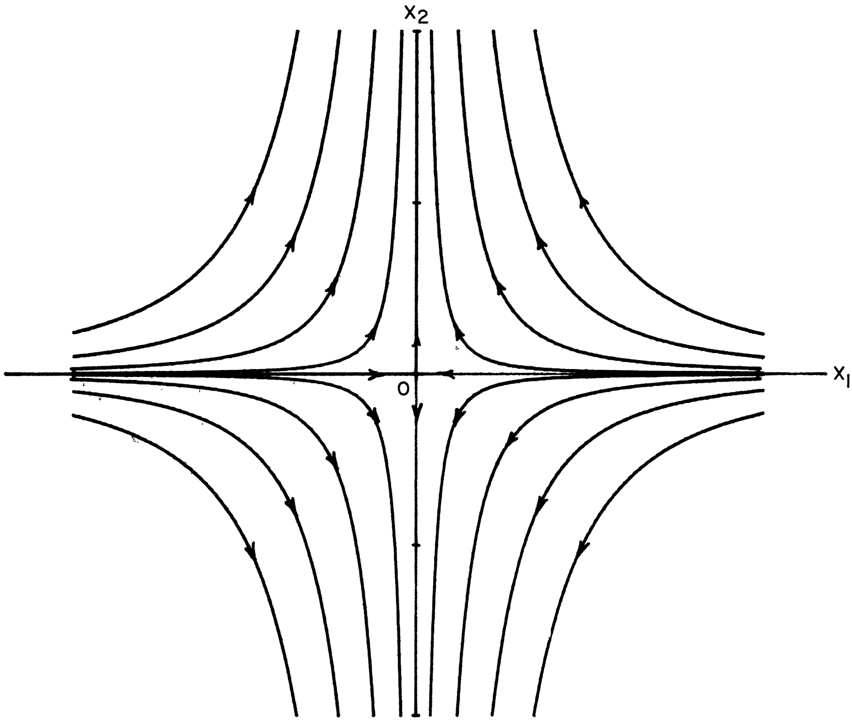


**Figure 1**

on the right-hand side of (1) defines a mapping $\mathbf{f}\colon \mathbf{R}^2 \to \mathbf{R}^2$ (linear in this case). This mapping (which need not be linear) defines a *vector field on* $\mathbf{R}^2$; i.e., to each point $\mathbf{x} \in \mathbf{R}^2$, the mapping $\mathbf{f}$ assigns a vector $\mathbf{f}(\mathbf{x})$. If we draw each vector $\mathbf{f}(\mathbf{x})$ with its initial point at the point $\mathbf{x} \in \mathbf{R}^2$, we obtain a geometrical representation of the vector field as shown in Figure 2. Note that at each point $\mathbf{x}$ in the phase space $\mathbf{R}^2$, the solution curves (2) are tangent to the vectors in the vector field $A\mathbf{x}$. This follows since at time $t = t_0$, the velocity vector $\mathbf{v}_0 = \dot{\mathbf{x}}(t_0)$ is tangent to the curve $\mathbf{x} = \mathbf{x}(t)$ at the point $\mathbf{x}_0 = \mathbf{x}(t_0)$ and since $\dot{\mathbf{x}} = A\mathbf{x}$ along the solution curves.

Consider the following uncoupled linear system in $\mathbf{R}^3$:

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 \\ \dot{x}_3 &= -x_3 \end{aligned} \tag{3}$$

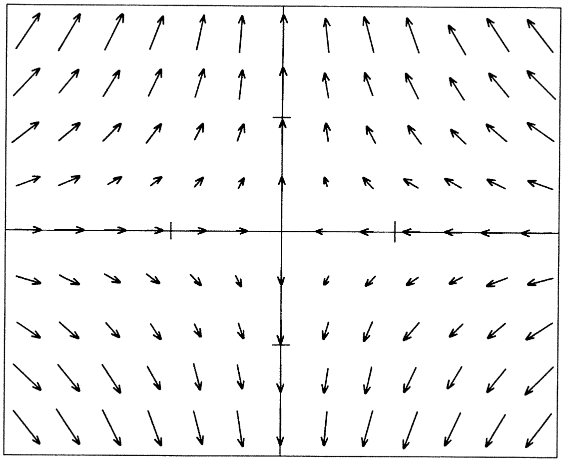

**Figure 2**

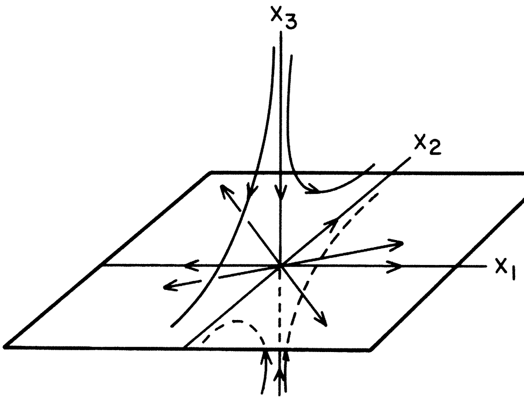


**Figure 3**

The general solution is given by

$$x_1(t) = c_1 e^t$$
$$x_2(t) = c_2 e^t$$
$$x_3(t) = c_3 e^{-t}.$$

And the phase portrait for this system is shown in Figure 3 above. The $x_1, x_2$ plane is referred to as the *unstable subspace* of the system (3) and the $x_3$ axis is called the *stable subspace* of the system (3). Precise definitions of the stable and unstable subspaces of a linear system will be given in the next section.

## Problem Set 1

1. Find the general solution and draw the phase portrait for the following linear systems:

(a) $$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= x_2\end{aligned}$$

(b) $$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= 2x_2\end{aligned}$$

(c) $$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= 3x_2\end{aligned}$$

(d) $$\begin{aligned}\dot{x}_1 &= -x_2\\ \dot{x}_2 &= x_1\end{aligned}$$

(e) $$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= -x_2\end{aligned}$$

**Hint:** Write (d) as a second-order linear differential equation with constant coefficients, solve it by standard methods, and note that $x_1^2 + x_2^2 =$ constant on the solution curves. In (e), find $x_2(t) = c_2 e^{-t}$ and then the $x_1$-equation becomes a first order linear differential equation.

2. Find the general solution and draw the phase portraits for the following three-dimensional linear systems:

(a) $$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= x_2\\ \dot{x}_3 &= x_3\end{aligned}$$

(b) $$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= -x_2\\ \dot{x}_3 &= x_3\end{aligned}$$

(c) $$\begin{aligned}\dot{x}_1 &= -x_2\\ \dot{x}_2 &= x_1\\ \dot{x}_3 &= -x_3\end{aligned}$$

**Hint:** In (c), show that the solution curves lie on right circular cylinders perpendicular to the $x_1, x_2$ plane. Identify the stable and unstable subspaces in (a) and (b). The $x_3$-axis is the stable subspace in (c) and the $x_1, x_2$ plane is called the center subspace in (c); cf. Section 1.9.

3. Find the general solution of the linear system

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = ax_2$$

where $a$ is a constant. Sketch the phase portraits for $a = -1$, $a = 0$ and $a = 1$ and notice that the qualitative structure of the phase portrait is the same for all $a < 0$ as well as for all $a > 0$, but that it changes at the parameter value $a = 0$ called a bifurcation value.

4. Find the general solution of the linear system (1) when $A$ is the $n \times n$ diagonal matrix $A = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. What condition on the eigenvalues $\lambda_1, \ldots, \lambda_n$ will guarantee that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions $\mathbf{x}(t)$ of (1)?

5. What is the relationship between the vector fields defined by

$$\dot{\mathbf{x}} = A\mathbf{x}$$

and

$$\dot{\mathbf{x}} = kA\mathbf{x}$$

where $k$ is a non-zero constant? (Describe this relationship both for $k$ positive and $k$ negative.)

6. (a) If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions of the linear system (1), prove that for any constants $a$ and $b$, $\mathbf{w}(t) = a\mathbf{u}(t) + b\mathbf{v}(t)$ is a solution.

   (b) For

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},$$

   find solutions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ of $\dot{\mathbf{x}} = A\mathbf{x}$ such that every solution is a linear combination of $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

## 1.2 Diagonalization

The algebraic technique of diagonalizing a square matrix $A$ can be used to reduce the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

to an uncoupled linear system. We first consider the case when $A$ has real, distinct eigenvalues. The following theorem from linear algebra then allows us to solve the linear system (1).

**Theorem.** *If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix $A$ are real and distinct, then any set of corresponding eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ forms a basis for $\mathbf{R}^n$, the matrix $P = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$ is invertible and*

$$P^{-1}AP = \text{diag}[\lambda_1, \ldots, \lambda_n].$$

This theorem says that if a linear transformation $T: \mathbf{R}^n \to \mathbf{R}^n$ is represented by the $n \times n$ matrix $A$ with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for $\mathbf{R}^n$, then with respect to any basis of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, $T$ is represented by the diagonal matrix of eigenvalues, $\operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. A proof of this theorem can be found, for example, in Lowenthal [Lo].

In order to reduce the system (1) to an uncoupled linear system using the above theorem, define the linear transformation of coordinates

$$\mathbf{y} = P^{-1}\mathbf{x}$$

where $P$ is the invertible matrix defined in the theorem. Then

$$\mathbf{x} = P\mathbf{y},$$
$$\dot{\mathbf{y}} = P^{-1}\dot{\mathbf{x}} = P^{-1}A\mathbf{x} = P^{-1}AP\mathbf{y}$$

and, according to the above theorem, we obtain the uncoupled linear system

$$\dot{\mathbf{y}} = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]\mathbf{y}.$$

This uncoupled linear system has the solution

$$\mathbf{y}(t) = \operatorname{diag}\left[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right]\mathbf{y}(0).$$

(Cf. problem 4 in Problem Set 1.) And then since $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$ and $\mathbf{x}(t) = P\mathbf{y}(t)$, it follows that (1) has the solution

$$\mathbf{x}(t) = PE(t)P^{-1}\mathbf{x}(0). \tag{2}$$

where $E(t)$ is the diagonal matrix

$$E(t) = \operatorname{diag}\left[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right].$$

**Corollary.** *Under the hypotheses of the above theorem, the solution of the linear system* (1) *is given by the function* $\mathbf{x}(t)$ *defined by* (2).

**Example.** Consider the linear system

$$\dot{x}_1 = -x_1 - 3x_2$$
$$\dot{x}_2 = 2x_2$$

which can be written in the form (1) with the matrix

$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}.$$

The eigenvalues of $A$ are $\lambda_1 = -1$ and $\lambda_2 = 2$. A pair of corresponding eigenvectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The matrix $P$ and its inverse are then given by

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The student should verify that

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then under the coordinate transformation $\mathbf{y} = P^{-1}\mathbf{x}$, we obtain the uncoupled linear system

$$\begin{aligned} \dot{y}_1 &= -y_1 \\ \dot{y}_2 &= 2y_2 \end{aligned}$$

which has the general solution $y_1(t) = c_1 e^{-t}$, $y_2(t) = c_2 e^{2t}$. The phase portrait for this system is given in Figure 1 in Section 1.1 which is reproduced below. And according to the above corollary, the general solution to the original linear system of this example is given by

$$\mathbf{x}(t) = P \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1}\mathbf{c}$$

where $\mathbf{c} = \mathbf{x}(0)$, or equivalently by

$$\begin{aligned} x_1(t) &= c_1 e^{-t} + c_2(e^{-t} - e^{2t}) \\ x_2(t) &= c_2 e^{2t}. \end{aligned} \tag{4}$$

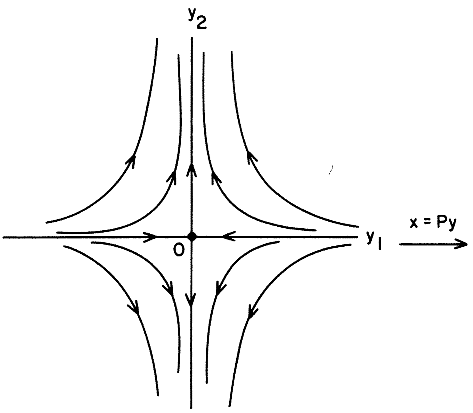


**Figure 1**

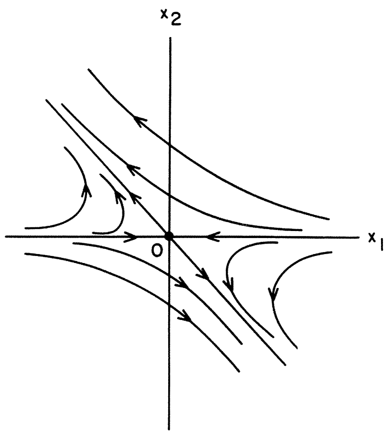


**Figure 2**

The phase portrait for the linear system of this example can be found by sketching the solution curves defined by (4). It is shown in Figure 2. The phase portrait in Figure 2 can also be obtained from the phase portrait in Figure 1 by applying the linear transformation of coordinates $\mathbf{x} = P\mathbf{y}$. Note that the subspaces spanned by the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the matrix $A$ determine the stable and unstable subspaces of the linear system (1) according to the following definition:

Suppose that the $n \times n$ matrix $A$ has $k$ negative eigenvalues $\lambda_1, \ldots, \lambda_k$ and $n - k$ positive eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ and that these eigenvalues are distinct. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a corresponding set of eigenvectors. Then the *stable and unstable subspaces of the linear system* (1), $E^s$ *and* $E^u$, are the linear subspaces spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ respectively; i.e.,

$$E^s = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$$
$$E^u = \operatorname{Span}\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}.$$

If the matrix $A$ has pure imaginary eigenvalues, then there is also a center subspace $E^c$; cf. Problem 2(c) in Section 1.1. The stable, unstable and center subspaces are defined for the general case in Section 1.9.

## PROBLEM SET 2

1. Find the eigenvalues and eigenvectors of the matrix $A$ and show that $B = P^{-1}AP$ is a diagonal matrix. Solve the linear system $\dot{\mathbf{y}} = B\mathbf{y}$ and then solve $\dot{\mathbf{x}} = A\mathbf{x}$ using the above corollary. And then sketch the phase portraits in both the $\mathbf{x}$ plane and $\mathbf{y}$ plane.

   (a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

2. Find the eigenvalues and eigenvectors for the matrix $A$, solve the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, determine the stable and unstable subspaces for the linear system, and sketch the phase portrait for

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \mathbf{x}.$$

3. Write the following linear differential equations with constant coefficients in the form of the linear system (1) and solve:

(a) $\ddot{x} + \dot{x} - 2x = 0$

(b) $\ddot{x} + x = 0$

(c) $\dddot{x} - 2\ddot{x} - \dot{x} + 2x = 0$

**Hint:** Let $x_1 = x$, $x_2 = \dot{x}_1$, etc.

4. Using the corollary of this section solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

(a) with $A$ given by 1(a) above and $\mathbf{x}_0 = (1,2)^T$

(b) with $A$ given in problem 2 above and $\mathbf{x}_0 = (1,2,3)^T$.

5. Let the $n \times n$ matrix $A$ have real, distinct eigenvalues. Find conditions on the eigenvalues that are necessary and sufficient for $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ where $\mathbf{x}(t)$ is any solution of $\dot{\mathbf{x}} = A\mathbf{x}$.

6. Let the $n \times n$ matrix $A$ have real, distinct eigenvalues. Let $\phi(t, \mathbf{x}_0)$ be the solution of the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$
$$\mathbf{x}(0) = \mathbf{x}_0.$$

Show that for each fixed $t \in \mathbf{R}$,

$$\lim_{\mathbf{y}_0 \to \mathbf{x}_0} \phi(t, \mathbf{y}_0) = \phi(t, \mathbf{x}_0).$$

This shows that the solution $\phi(t, \mathbf{x}_0)$ is a continuous function of the initial condition.

7. Let the $2\times 2$ matrix $A$ have real, distinct eigenvalues $\lambda$ and $\mu$. Suppose that an eigenvector of $\lambda$ is $(1,0)^T$ and an eigenvector of $\mu$ is $(-1,1)^T$. Sketch the phase portraits of $\dot{\mathbf{x}} = A\mathbf{x}$ for the following cases:

(a) $0 < \lambda < \mu$ (b) $0 < \mu < \lambda$ (c) $\lambda < \mu < 0$

(d) $\lambda < 0 < \mu$ (e) $\mu < 0 < \lambda$ (f) $\lambda = 0$, $\mu > 0$.

## 1.3 Exponentials of Operators

In order to define the exponential of a linear operator $T: \mathbf{R}^n \to \mathbf{R}^n$, it is necessary to define the concept of convergence in the linear space $L(\mathbf{R}^n)$ of linear operators on $\mathbf{R}^n$. This is done using the *operator norm of* $T$ defined by

$$\|T\| = \max_{|\mathbf{x}| \le 1} |T(\mathbf{x})|$$

where $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x} \in \mathbf{R}^n$; i.e.,

$$|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

The operator norm has all of the usual properties of a norm, namely, for $S, T \in L(\mathbf{R}^n)$

(a) $\|T\| \geq 0$ and $\|T\| = 0$ iff $T = 0$

(b) $\|kT\| = |k|\,\|T\|$ for $k \in \mathbf{R}$

(c) $\|S + T\| \leq \|S\| + \|T\|$.

It follows from the Cauchy–Schwarz inequality that if $T \in L(\mathbf{R}^n)$ is represented by the matrix $A$ with respect to the standard basis for $\mathbf{R}^n$, then $\|A\| \leq \sqrt{n}\ell$ where $\ell$ is the maximum length of the rows of $A$.

The convergence of a sequence of operators $T_k \in L(\mathbf{R}^n)$ is then defined in terms of the operator norm as follows:

**Definition 1.** A sequence of linear operators $T_k \in L(\mathbf{R}^n)$ is said to converge to a linear operator $T \in L(\mathbf{R}^n)$ as $k \to \infty$, i.e.,

$$\lim_{k \to \infty} T_k = T,$$

if for all $\varepsilon > 0$ there exists an $N$ such that for $k \geq N$, $\|T - T_k\| < \varepsilon$.

**Lemma.** *For $S, T \in L(\mathbf{R}^n)$ and $\mathbf{x} \in \mathbf{R}^n$,*

(1) $|T(\mathbf{x})| \leq \|T\|\,|\mathbf{x}|$

(2) $\|TS\| \leq \|T\|\,\|S\|$

(3) $\|T^k\| \leq \|T\|^k$ *for* $k = 0, 1, 2, \ldots.$

**Proof.** (1) is obviously true for $\mathbf{x} = \mathbf{0}$. For $\mathbf{x} \neq \mathbf{0}$ define the unit vector $\mathbf{y} = \mathbf{x}/|\mathbf{x}|$. Then from the definition of the operator norm,

$$\|T\| \geq |T(\mathbf{y})| = \frac{1}{|\mathbf{x}|}|T(\mathbf{x})|.$$

(2) For $|\mathbf{x}| \leq 1$, it follows from (1) that

$$\begin{aligned} |T(S(\mathbf{x}))| &\leq \|T\|\,|S(\mathbf{x})| \\ &\leq \|T\|\,\|S\|\,|\mathbf{x}| \\ &\leq \|T\|\,\|S\|. \end{aligned}$$

Therefore,

$$\|TS\| = \max_{|\mathbf{x}| \leq 1} |TS(\mathbf{x})| \leq \|T\|\,\|S\|$$

and (3) is an immediate consequence of (2).

**Theorem.** *Given $T \in L(\mathbf{R}^n)$ and $t_0 > 0$, the series*

$$\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$$

*is absolutely and uniformly convergent for all $|t| \leq t_0$.*

**Proof.** Let $\|T\| = a$. It then follows from the above lemma that for $|t| \leq t_0$,

$$\left\| \frac{T^k t^k}{k!} \right\| \leq \frac{\|T\|^k |t|^k}{k!} \leq \frac{a^k t_0^k}{k!}.$$

But

$$\sum_{k=0}^{\infty} \frac{a^k t_0^k}{k!} = e^{a t_0}.$$

It therefore follows from the Weierstrass M-Test that the series

$$\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$$

is absolutely and uniformly convergent for all $|t| \leq t_0$; cf. [R], p. 148.

The exponential of the linear operator $T$ is then defined by the absolutely convergent series

$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}.$$

It follows from properties of limits that $e^T$ is a linear operator on $\mathbf{R}^n$ and it follows as in the proof of the above theorem that $\|e^T\| \leq e^{\|T\|}$.

Since our main interest in this chapter is the solution of linear systems of the form

$$\dot{\mathbf{x}} = A\mathbf{x},$$

we shall assume that the linear transformation $T$ on $\mathbf{R}^n$ is represented by the $n \times n$ matrix $A$ with respect to the standard basis for $\mathbf{R}^n$ and define the exponential $e^{At}$.

**Definition 2.** Let $A$ be an $n \times n$ matrix. Then for $t \in \mathbf{R}$,

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

For an $n \times n$ matrix $A$, $e^{At}$ is an $n \times n$ matrix which can be computed in terms of the eigenvalues and eigenvectors of $A$. This will be carried out in the remainder of this chapter. As in the proof of the above theorem $\|e^{At}\| \leq e^{\|A\||t|}$ where $\|A\| = \|T\|$ and $T$ is the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

We next establish some basic properties of the linear transformation $e^T$ in order to facilitate the computation of $e^T$ or of the $n \times n$ matrix $e^A$.

**Proposition 1.** *If $P$ and $T$ are linear transformations on $\mathbf{R}^n$ and $S = PTP^{-1}$, then $e^S = Pe^TP^{-1}$.*

**Proof.** It follows from the definition of $e^S$ that

$$e^S = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(PTP^{-1})^k}{k!} = P \lim_{n\to\infty} \sum_{k=0}^{n} \frac{T^k}{k!} P^{-1} = Pe^TP^{-1}.$$

The next result follows directly from Proposition 1 and Definition 2.

**Corollary 1.** *If $P^{-1}AP = \operatorname{diag}[\lambda_j]$ then $e^{At} = P\operatorname{diag}[e^{\lambda_j t}]P^{-1}$.*

**Proposition 2.** *If $S$ and $T$ are linear transformations on $\mathbf{R}^n$ which commute, i.e., which satisfy $ST = TS$, then $e^{S+T} = e^Se^T$.*

**Proof.** If $ST = TS$, then by the binomial theorem

$$(S+T)^n = n! \sum_{j+k=n} \frac{S^jT^k}{j!k!}.$$

Therefore,

$$e^{S+T} = \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{S^jT^k}{j!k!} = \sum_{j=0}^{\infty} \frac{S^j}{j!} \sum_{k=0}^{\infty} \frac{T^k}{k!} = e^Se^T.$$

We have used the fact that the product of two absolutely convergent series is an absolutely convergent series which is given by its Cauchy product; cf. [R], p. 74.

Upon setting $S = -T$ in Proposition 2, we obtain

**Corollary 2.** *If $T$ is a linear transformation on $\mathbf{R}^n$, the inverse of the linear transformation $e^T$ is given by $(e^T)^{-1} = e^{-T}$.*

**Corollary 3.** *If*

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

*then*

$$e^A = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}.$$

**Proof.** If $\lambda = a + ib$, it follows by induction that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k = \begin{bmatrix} \operatorname{Re}(\lambda^k) & -\operatorname{Im}(\lambda^k) \\ \operatorname{Im}(\lambda^k) & \operatorname{Re}(\lambda^k) \end{bmatrix}$$

where Re and Im denote the real and imaginary parts of the complex number $\lambda$ respectively. Thus,

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \begin{bmatrix} \operatorname{Re}(\frac{\lambda^k}{k!}) & -\operatorname{Im}(\frac{\lambda^k}{k!}) \\ \operatorname{Im}(\frac{\lambda^k}{k!}) & \operatorname{Re}(\frac{\lambda^k}{k!}) \end{bmatrix} \\ &= \begin{bmatrix} \operatorname{Re}(e^\lambda) & -\operatorname{Im}(e^\lambda) \\ \operatorname{Im}(e^\lambda) & \operatorname{Re}(e^\lambda) \end{bmatrix} \\ &= e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}. \end{aligned}$$

Note that if $a = 0$ in Corollary 3, then $e^A$ is simply a rotation through $b$ radians.

**Corollary 4.** *If*

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

*then*

$$e^A = e^a \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

**Proof.** Write $A = aI + B$ where

$$B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}.$$

Then $aI$ commutes with $B$ and by Proposition 2,

$$e^A = e^{aI}e^B = e^a e^B.$$

And from the definition

$$e^B = I + B + B^2/2! + \cdots = I + B$$

since by direct computation $B^2 = B^3 = \cdots = 0$.

We can now compute the matrix $e^{At}$ for any $2\times2$ matrix $A$. In Section 1.8 of this chapter it is shown that there is an invertible $2 \times 2$ matrix $P$ (whose columns consist of generalized eigenvectors of $A$) such that the matrix

$$B = P^{-1}AP$$

has one of the following forms

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{or} \quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

It then follows from the above corollaries and Definition 2 that

$$e^{Bt} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}, \quad e^{Bt} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad e^{Bt} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

respectively. And by Proposition 1, the matrix $e^{At}$ is then given by

$$e^{At} = P e^{Bt} P^{-1}.$$

As we shall see in Section 1.4, finding the matrix $e^{At}$ is equivalent to solving the linear system (1) in Section 1.1.

## Problem Set 3

1. Compute the operator norm of the linear transformation defined by the following matrices:

   (a) $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$.

   **Hint:** In (c) maximize $|A\mathbf{x}|^2 = 26x_1^2 + 10x_1x_2 + x_2^2$ subject to the constraint $x_1^2 + x_2^2 = 1$ and use the result of Problem 2; or use the fact that $\|A\| = [\text{Max eigenvalue of } A^T A]^{1/2}$.

2. Show that the operator norm of a linear transformation $T$ on $\mathbf{R}^n$ satisfies

$$\|T\| = \max_{|\mathbf{x}|=1} |T(\mathbf{x})| = \sup_{\mathbf{x}\neq 0} \frac{|T(\mathbf{x})|}{|\mathbf{x}|}.$$

3. Use the lemma in this section to show that if $T$ is an invertible linear transformation then $\|T\| > 0$ and

$$\|T^{-1}\| \geq \frac{1}{\|T\|}.$$

4. If $T$ is a linear transformation on $\mathbf{R}^n$ with $\|T - I\| < 1$, prove that $T$ is invertible and that the series $\sum_{k=0}^{\infty}(I - T)^k$ converges absolutely to $T^{-1}$.

   **Hint:** Use the geometric series.

5. Compute the exponentials of the following matrices:

   (a) $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

6. (a) For each matrix in Problem 5 find the eigenvalues of $e^A$.

   (b) Show that if $\mathbf{x}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$, then $\mathbf{x}$ is also an eigenvector of $e^A$ corresponding to the eigenvalue $e^\lambda$.

   (c) If $A = P\operatorname{diag}[\lambda_j]P^{-1}$, use Corollary 1 to show that

$$\det e^A = e^{\operatorname{trace} A}.$$

   Also, using the results in the last paragraph of this section, show that this formula holds for any $2 \times 2$ matrix $A$.

7. Compute the exponentials of the following matrices:

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

   **Hint:** Write the matrices in (b) and (c) as a diagonal matrix $S$ plus a matrix $N$. Show that $S$ and $N$ commute and compute $e^S$ as in part (a) and $e^N$ by using the definition.

8. Find $2 \times 2$ matrices $A$ and $B$ such that $e^{A+B} \neq e^A e^B$.

9. Let $T$ be a linear operator on $\mathbf{R}^n$ that leaves a subspace $E \subset \mathbf{R}^n$ invariant; i.e., for all $\mathbf{x} \in E$, $T(\mathbf{x}) \in E$. Show that $e^T$ also leaves $E$ invariant.

## 1.4 The Fundamental Theorem for Linear Systems

Let $A$ be an $n \times n$ matrix. In this section we establish the fundamental fact that for $\mathbf{x}_0 \in \mathbf{R}^n$ the initial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned} \tag{1}$$

has a unique solution for all $t \in \mathbf{R}$ which is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0. \tag{2}$$

Notice the similarity in the form of the solution (2) and the solution $x(t) = e^{at}x_0$ of the elementary first-order differential equation $\dot{x} = ax$ and initial condition $x(0) = x_0$.

In order to prove this theorem, we first compute the derivative of the exponential function $e^{At}$ using the basic fact from analysis that two convergent limit processes can be interchanged if one of them converges uniformly. This is referred to as Moore's Theorem; cf. Graves [G], p. 100 or Rudin [R], p. 149.

**Lemma.** *Let $A$ be a square matrix, then*

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

**Proof.** Since $A$ commutes with itself, it follows from Proposition 2 and Definition 2 in Section 3 that

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \lim_{h\to 0} \frac{e^{A(t+h)} - e^{At}}{h} \\
&= \lim_{h\to 0} e^{At} \frac{(e^{Ah} - I)}{h} \\
&= e^{At} \lim_{h\to 0} \lim_{k\to\infty} \left( A + \frac{A^2 h}{2!} + \cdots + \frac{A^k h^{k-1}}{k!} \right) \\
&= Ae^{At}.
\end{aligned}$$

The last equality follows since by the theorem in Section 1.3 the series defining $e^{Ah}$ converges uniformly for $|h| \le 1$ and we can therefore interchange the two limits.

**Theorem (The Fundamental Theorem for Linear Systems).** *Let $A$ be an $n \times n$ matrix. Then for a given $\mathbf{x}_0 \in \mathbf{R}^n$, the initial value problem*

$$\begin{aligned}
\dot{\mathbf{x}} &= A\mathbf{x} \\
\mathbf{x}(0) &= \mathbf{x}_0
\end{aligned} \tag{1}$$

*has a unique solution given by*

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0. \tag{2}$$

**Proof.** By the preceding lemma, if $\mathbf{x}(t) = e^{At}\mathbf{x}_0$, then

$$\mathbf{x}'(t) = \frac{d}{dt}e^{At}\mathbf{x}_0 = Ae^{At}\mathbf{x}_0 = A\mathbf{x}(t)$$

for all $t \in \mathbf{R}$. Also, $\mathbf{x}(0) = I\mathbf{x}_0 = \mathbf{x}_0$. Thus $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ is a solution. To see that this is the only solution, let $\mathbf{x}(t)$ be any solution of the initial value problem (1) and set

$$\mathbf{y}(t) = e^{-At}\mathbf{x}(t).$$

Then from the above lemma and the fact that $\mathbf{x}(t)$ is a solution of (1)

$$\begin{aligned}\mathbf{y}'(t) &= -Ae^{-At}\mathbf{x}(t) + e^{-At}\mathbf{x}'(t)\\ &= -Ae^{-At}\mathbf{x}(t) + e^{-At}A\mathbf{x}(t)\\ &= \mathbf{0}\end{aligned}$$

for all $t \in \mathbf{R}$ since $e^{-At}$ and $A$ commute. Thus, $\mathbf{y}(t)$ is a constant. Setting $t = 0$ shows that $\mathbf{y}(t) = \mathbf{x}_0$ and therefore any solution of the initial value problem (1) is given by $\mathbf{x}(t) = e^{At}\mathbf{y}(t) = e^{At}\mathbf{x}_0$. This completes the proof of the theorem.

**Example.** Solve the initial value problem

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x}\\ \mathbf{x}(0) &= \begin{bmatrix}1\\0\end{bmatrix}\end{aligned}$$

for

$$A = \begin{bmatrix}-2 & -1\\ 1 & -2\end{bmatrix}$$

and sketch the solution curve in the phase plane $\mathbf{R}^2$. By the above theorem and Corollary 3 of the last section, the solution is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{-2t}\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = e^{-2t}\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}.$$

It follows that $|\mathbf{x}(t)| = e^{-2t}$ and that the angle $\theta(t) = \tan^{-1} x_2(t)/x_1(t) = t$. The solution curve therefore spirals into the origin as shown in Figure 1 below.

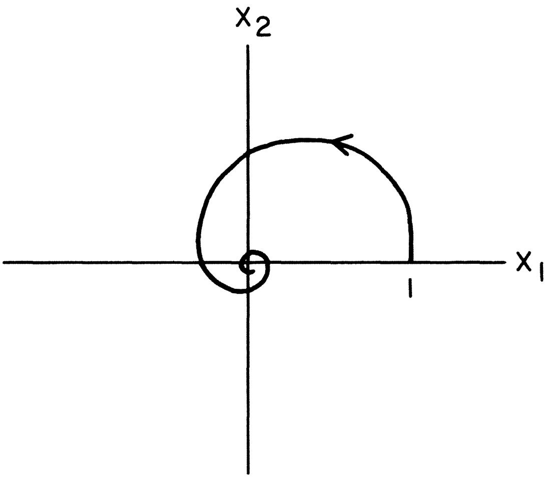


**Figure 1**

## Problem Set 4

1. Use the forms of the matrix $e^{Bt}$ computed in Section 1.3 and the theorem in this section to solve the linear system $\dot{\mathbf{x}} = B\mathbf{x}$ for

   (a) $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$

   (b) $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

   (c) $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

2. Solve the following linear system and sketch its phase portrait

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x}.$$

   The origin is called a stable focus for this system.

3. Find $e^{At}$ and solve the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ for

   (a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

   Cf. Problem 1 in Problem Set 2.

4. Given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

   Compute the $3 \times 3$ matrix $e^{At}$ and solve $\dot{\mathbf{x}} = A\mathbf{x}$. Cf. Problem 2 in Problem Set 2.

5. Find the solution of the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ where

   (a) $A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (d) $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}$.

6. Let $T$ be a linear transformation on $\mathbf{R}^n$ that leaves a subspace $E \subset \mathbf{R}^n$ invariant (i.e., for all $\mathbf{x} \in E$, $T(\mathbf{x}) \in E$) and let $T(\mathbf{x}) = A\mathbf{x}$ with respect to the standard basis for $\mathbf{R}^n$. Show that if $\mathbf{x}(t)$ is the solution of the initial value problem

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x}\\ \mathbf{x}(0) &= \mathbf{x}_0\end{aligned}$$

with $\mathbf{x}_0 \in E$, then $\mathbf{x}(t) \in E$ for all $t \in \mathbf{R}$.

7. Suppose that the square matrix $A$ has a negative eigenvalue. Show that the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ has at least one nontrivial solution $\mathbf{x}(t)$ that satisfies

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}.$$

8. (Continuity with respect to initial conditions.) Let $\phi(t, \mathbf{x}_0)$ be the solution of the initial value problem (1). Use the Fundamental Theorem to show that for each fixed $t \in \mathbf{R}$

$$\lim_{\mathbf{y}\to\mathbf{x}_0} \phi(t, \mathbf{y}) = \phi(t, \mathbf{x}_0).$$

## 1.5 Linear Systems in $\mathrm{R}^2$

In this section we discuss the various phase portraits that are possible for the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

when $\mathbf{x} \in \mathbf{R}^2$ and $A$ is a $2 \times 2$ matrix. We begin by describing the phase portraits for the linear system

$$\dot{\mathbf{x}} = B\mathbf{x} \tag{2}$$

where the matrix $B = P^{-1}AP$ has one of the forms given at the end of Section 1.3. The phase portrait for the linear system (1) above is then obtained from the phase portrait for (2) under the linear transformation of coordinates $\mathbf{x} = P\mathbf{y}$ as in Figures 1 and 2 in Section 1.2.

First of all, if

$$B = \begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}, \quad B = \begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}, \quad \text{or} \quad B = \begin{bmatrix}a & -b\\ b & a\end{bmatrix},$$

it follows from the fundamental theorem in Section 1.4 and the form of the matrix $e^{Bt}$ computed in Section 1.3 that the solution of the initial value problem (2) with $\mathbf{x}(0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = \begin{bmatrix}e^{\lambda t} & 0\\ 0 & e^{\mu t}\end{bmatrix}\mathbf{x}_0, \quad \mathbf{x}(t) = e^{\lambda t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\mathbf{x}_0,$$

or

$$\mathbf{x}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{x}_0$$

respectively. We now list the various phase portraits that result from these solutions, grouped according to their topological type with a finer classification of sources and sinks into various types of unstable and stable nodes and foci:

**Case I.** $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ with $\lambda < 0 < \mu$.

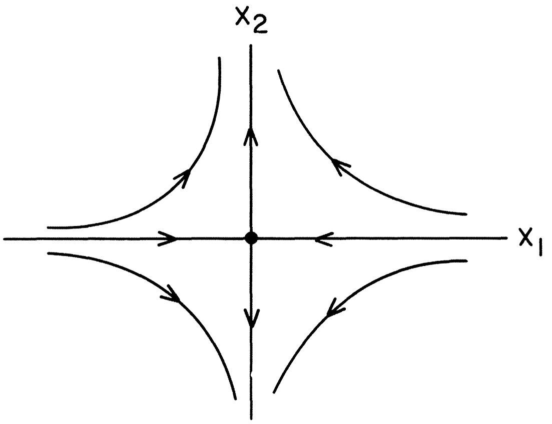


**Figure 1.** A saddle at the origin.

The phase portrait for the linear system (2) in this case is given in Figure 1. See the first example in Section 1.1. The system (2) is said to have a *saddle at the origin* in this case. If $\mu < 0 < \lambda$, the arrows in Figure 1 are reversed. Whenever $A$ has two real eigenvalues of opposite sign, $\lambda < 0 < \mu$, the phase portrait for the linear system (1) is linearly equivalent to the phase portrait shown in Figure 1; i.e., it is obtained from Figure 1 by a linear transformation of coordinates; and the stable and unstable subspaces of (1) are determined by the eigenvectors of $A$ as in the Example in Section 1.2. The four non-zero trajectories or solution curves that approach the equilibrium point at the origin as $t \to \pm\infty$ are called *separatrices* of the system.

**Case II.** $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ with $\lambda \le \mu < 0$ or $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ with $\lambda < 0$.

The phase portraits for the linear system (2) in these cases are given in Figure 2. Cf. the phase portraits in Problems 1(a), (b) and (c) of Problem Set 1 respectively. The origin is referred to as a *stable node* in each of these cases. It is called a proper node in the first case with $\lambda = \mu$ and an improper node in the other two cases. If $\lambda \geq \mu > 0$ or if $\lambda > 0$ in Case II, the arrows in Figure 2 are reversed and the origin is referred to as an unstable node. Whenever $A$ has two negative eigenvalues $\lambda \leq \mu < 0$, the phase portrait of the linear system (1) is linearly equivalent to one of the phase portraits shown in Figure 2. The stability of the node is determined by the sign of the eigenvalues: *stable* if $\lambda \leq \mu < 0$ and *unstable* if $\lambda \geq \mu > 0$. Note that each trajectory in Figure 2 approaches the equilibrium point at the origin along a well-defined tangent line $\theta = \theta_0$, determined by an eigenvector of $A$, as $t \to \infty$.

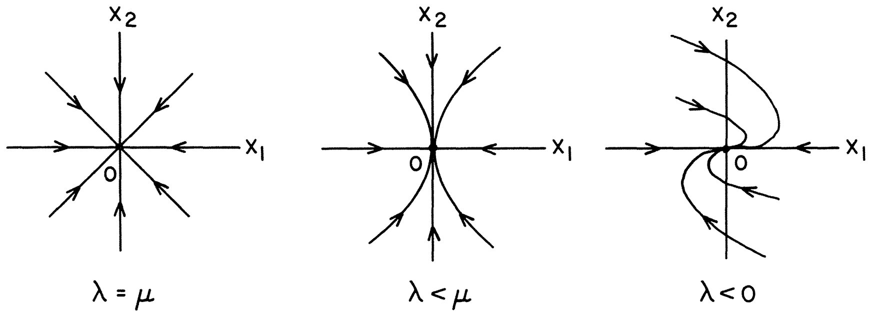


**Figure 2.** A stable node at the origin.

**Case III.** $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a < 0$.

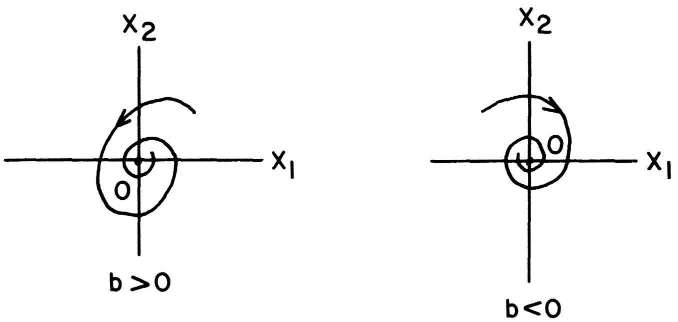


**Figure 3.** A stable focus at the origin.

The phase portrait for the linear system (2) in this case is given in Figure 3. Cf. Problem 9. The origin is referred to as a *stable focus* in these cases. If $a > 0$, the trajectories spiral away from the origin with increasing

$t$ and the origin is called an unstable focus. Whenever $A$ has a pair of complex conjugate eigenvalues with nonzero real part, $a \pm ib$, with $a < 0$, the phase portraits for the system (1) is linearly equivalent to one of the phase portraits shown in Figure 3. Note that the trajectories in Figure 3 do not approach the origin along well-defined tangent lines; i.e., the angle $\theta(t)$ that the vector $\mathbf{x}(t)$ makes with the $x_1$-axis does not approach a constant $\theta_0$ as $t \to \infty$, but rather $|\theta(t)| \to \infty$ as $t \to \infty$ and $|\mathbf{x}(t)| \to 0$ as $t \to \infty$ in this case.

**Case IV.** $B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$

The phase portrait for the linear system (2) in this case is given in Figure 4. Cf. Problem 1(d) in Problem Set 1. The system (2) is said to have a *center at the origin* in this case. Whenever $A$ has a pair of pure imaginary complex conjugate eigenvalues, $\pm ib$, the phase portrait of the linear system (1) is linearly equivalent to one of the phase portraits shown in Figure 4. Note that the trajectories or solution curves in Figure 4 lie on circles $|\mathbf{x}(t)| =$ constant. In general, the trajectories of the system (1) will lie on ellipses and the solution $\mathbf{x}(t)$ of (1) will satisfy $m \leq |\mathbf{x}(t)| \leq M$ for all $t \in \mathbf{R}$; cf. the following Example. The angle $\theta(t)$ also satisfies $|\theta(t)| \to \infty$ as $t \to \infty$ in this case.

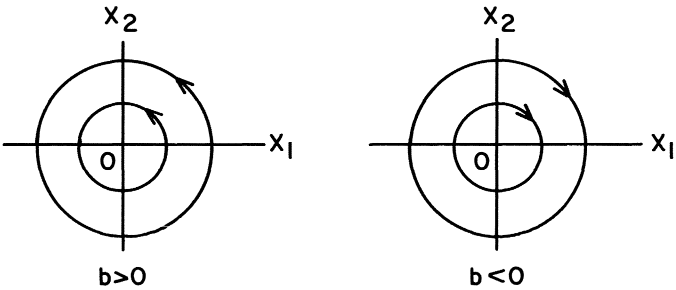


**Figure 4.** A center at the origin.

If one (or both) of the eigenvalues of $A$ is zero, i.e., if $\det A = 0$, the origin is called a *degenerate equilibrium point of* (1). The various portraits for the linear system (1) are determined in Problem 4 in this case.

**Example (A linear system with a center at the origin.)** The linear system

$$\dot{\mathbf{x}} = A\mathbf{x}$$

with

$$A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$$

has a center at the origin since the matrix $A$ has eigenvalues $\lambda = \pm 2i$. According to the theorem in Section 1.6, the invertible matrix

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{with} \quad P^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

reduces $A$ to the matrix

$$B = P^{-1}AP = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

The student should verify the calculation.

The solution to the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, as determined by Sections 1.3 and 1.4, is then given by

$$\mathbf{x}(t) = P \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} P^{-1}\mathbf{c} = \begin{bmatrix} \cos 2t & -2\sin 2t \\ 1/2 \sin 2t & \cos 2t \end{bmatrix} \mathbf{c}$$

where $\mathbf{c} = \mathbf{x}(0)$, or equivalently by

$$x_1(t) = c_1 \cos 2t - 2c_2 \sin 2t$$
$$x_2(t) = 1/2 c_1 \sin 2t + c_2 \cos 2t.$$

It is then easily shown that the solutions satisfy

$$x_1^2(t) + 4x_2^2(t) = c_1^2 + 4c_2^2$$

for all $t \in \mathbf{R}$; i.e., the trajectories of this system lie on ellipses as shown in Figure 5.

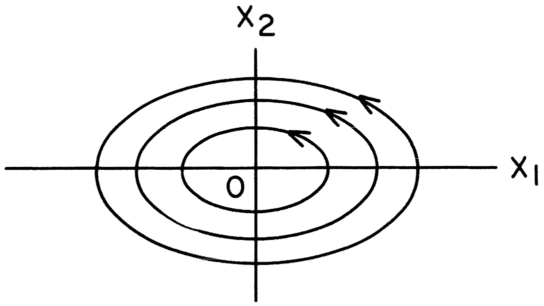


**Figure 5.** A center at the origin.

**Definition 1.** The linear system (1) is said to have *a saddle, a node, a focus* or *a center at the origin* if the matrix $A$ is similar to one of the matrices $B$ in Cases I, II, III or IV respectively. i.e., if its phase portrait