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CONTENTS

CHAPTER 1

NONLINEAR SYSTEMS: LOCAL THEORY

In Chapter 1 we saw that any linear system (**??**) has a unique solution through each point x_0 in the phase space $\mathbf{R}^n$; the solution is given by $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ and it is defined for all $t \in \mathbf{R}$. In this chapter we begin our study of nonlinear systems of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2}$$

where $\mathbf{f} : E \to \mathbf{R}^n$ and E is an open subset of $\mathbf{R}^n$. We show that under certain conditions on the function f, the nonlinear system (2) has a unique solution through each point $\mathrm{x}_0 \in E$ defined on a maximal interval of existence $(\alpha, \beta) \subset \mathbf{R}$. In general, it is not possible to solve the nonlinear system (2); however, a great deal of qualitative information about the local behavior of the solution is determined in this chapter. In particular, we establish the Hartman-Grobman Theorem and the Stable Manifold Theorem which show that topologically the local behavior of the nonlinear system (2) near an equilibrium point x_0 where $f(x_0) = 0$ is typically determined by the behavior of the linear system (**??**) near the origin when the matrix $A = D\mathrm{f}(\mathrm{x}_0)$, the derivative of $\mathbf{f}$ at $\mathbf{x}_0$. We also discuss some of the ramifications of these theorems for two-dimensional systems when $\det D\mathbf{f}(\mathrm{x}_0) \neq 0$ and cite some of the local results of Andronov et al. [A-I] for planar systems (2) with $\det D\mathbf{f}(\mathbf{x}_0) = 0$.

1.1 Some Preliminary Concepts and Definitions

Before beginning our discussion of the fundamental theory of nonlinear systems of differential equations, we present some preliminary concepts and definitions. First of all, in this book we shall only consider autonomous systems of ordinary differential equations (2) as opposed to nonautonomous systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{3}$$

where the function f can depend on the independent variable t; however, any nonautonomous system (3) with $\mathbf{x} \in \mathbf{R}^n$ can be written as an autonomous system (2) with $\mathbf{x} \in \mathbf{R}^{n+1}$ simply by letting $x_{n+1} = t$ and $\dot{x}_{n+1} = 1$. The fundamental theory for (2) and (3) does not differ significantly although it is possible to obtain the existence and uniqueness of solutions of (3) under slightly weaker hypotheses on $\mathbf{f}$ as a function of t; cf. for example Coddington and Levinson [C/L].
Notice that the existence of the solution of the elementary differential equation (2) is given by

$$x(t) = x(0) + \int_0^t f(s)ds$$

if $f(t)$ is integrable. And in general, the differential equations (2) or (3) will have a solution if the function f is continuous; cf. $[C/L]$, p. 6. However, continuity of the function $\mathbf{f}$ in (2) is not sufficient to guarantee uniqueness of the solution as the next example shows.

Example 1. The initial value problem

$$\dot{x} = 3x^{2/3}$$

$$x(0) = 0$$

has two different solutions through the point $(0, 0)$, namely

$$u(t) = t^3$$

and

$$v(t) \equiv 0$$

for all $t \in \mathbf{R}$. Clearly, each of these functions satisfies the differential equation for all $t \in \mathbf{R}$ as well as the initial condition $x(0) = 0$. (The first solution $u(t) = t^3$ can be obtained by the method of separation of variables). Notice that the function $f(x) = 3x^{2/3}$ is continuous at $x = 0$ but that it is not differentiable there.

Another feature of nonlinear systems that differs from linear systems is that even when the function f in (2) is defined and continuous for all $\mathbf{x} \in \mathbf{R^n}$, the solution $\mathbf{x}(t)$ may become unbounded at some finite time $t = \beta$; i.e., the solution may only exist on some proper subinterval $(\alpha, \beta) \subset \mathbf{R}$. This is illustrated by the next example.

Example 2. Consider the initial value problem

$$\dot{x} = x^2$$
$$x(0) = 1.$$

The solution, which can be found by the method of separation of variables, is given by

$$x(t) = \frac{1}{1-t}.$$

This solution is only defined for $t \in (-\infty, 1)$ and

$$\lim_{t\to 1^-} x(t) = \infty.$$

The interval $(-\infty, 1)$ is called the maximal interval of existence of the solution of this initial value problem. Notice that the function $x(t) = (1- t)^{-1}$ has another branch defined on the interval $(1, \infty)$; however, this branch is not considered as part of the solution of the initial value problem since the initial time $t = 0 \notin (1, \infty)$.

Before stating and proving the fundamental existence-uniqueness theorem for the nonlinear system (1), it is first necessary to define some terminology and notation concerning the derivative $D\mathbf{f}$ of a function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$.

Definition 1. The function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ is differentiable at $\mathbf{x}_0 \in \mathbf{R}^n$ if there is a linear transformation $D\mathbf{f}(\mathrm{x}_0) \in L(\mathbf{R^n})$ that satisfies

$$\lim_{|\mathbf{h}|\to 0} \frac{|\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)\,\mathbf{h}|}{|\mathbf{h}|} = 0$$

The linear transformation $D\mathbf{f}(\mathbf{x}_0)$ is called the derivative of $\mathbf{f}$ at $\mathbf{x}_0$.

Theorem 1. If $\mathbf{f} : \mathbf{R}^n \to \mathbf{R^n}$ is differentiable at $\mathbf{x}_0$, then the partial derivatives $\frac{\partial f_i}{\partial x_j}, i, j = 1, \ldots, n$, all exist at x_0 and for all $\mathbf{x} \in \mathbf{R}^n$,

$$D\mathbf{f}(\mathbf{x}_0)\,\mathbf{x} = \sum_{j=1}^{n} \frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{x}_0)\, x_j.$$

Thus, if $\mathbf{f}$ is a differentiable function, the derivative $D\mathbf{f}$ is given by the $n \times n$ Jacobian matrix

$$D\mathbf{f} = \left[\frac{\partial f_i}{\partial x_j}\right]$$

Definition 2. Suppose that V_1 and V_2 are two normed linear spaces with respective norms $\|\cdot\|_1$ and $\|\cdot\|_2$; i.e., V_1 and V_2 are linear spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ satisfying a-c in Section 1.3 of Chapter 1 . Then

$$\text{F}: V_1 \to V_2$$

is continuous at $\text{x}_0 \in V_1$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{x} \in V_1$ and $\|\mathbf{x} - \mathbf{x}_0\|_1 < \delta$ implies that

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}_0)\|_2 < \varepsilon.$$

And $\mathbf{F}$ is said to be continuous on the set $E \subset V_1$ if it is continuous at each point $\mathbf{x} \in E$. If $\mathbf{F}$ is continuous on $E \subset V_1$, we write $\mathbf{F} \in C(E)$.

Definition 3. Suppose that $\mathbf{f} : E \to \mathbf{R}^n$ is differentiable on E. Then $\mathbf{f} \in C^1(E)$ if the derivative $D\mathbf{f} : E \to L(\mathbf{R}^n)$ is continuous on E.

The next theorem, gives a simple test for deciding whether or not a function $\text{f} : E \to \mathbf{R}^n$ belongs to $C^1(E)$.

Theorem 2. Suppose that E is an open subset of $\mathbf{R^n}$ and that $\mathbf{f} : E \to \mathbf{R}^n$. Then $\text{f} \in C^1(E)$ iff the partial derivatives $\dfrac{\partial f_i}{\partial x_j}, i, j = 1, \ldots, n$, exist and are continuous on E.

Remark 1. For E an open subset of $\mathbf{R}^n$, the higher order derivatives $D^k\mathbf{f}(\mathbf{x}_0)$ of a function $\mathbf{f} : E \to \mathbf{R}^n$ are defined in a similar way and it can be shown that $\text{f} \in C^k(E)$ if and only if the partial derivatives

$$\frac{\partial^k f_i}{\partial x_{j_1} \cdots \partial x_{j_k}}$$

with $i, j_1, \ldots, j_k = 1, \ldots, n$, exist and are continuous on E. Furthermore, $D^2\mathbf{f}(\mathbf{x}_0) : E \times E \to \mathbf{R}^n$ and for $(\mathbf{x}, \mathbf{y}) \in E \times E$ we have

$$D^2\mathbf{f}(\mathbf{x}_0)(\mathbf{x}, \mathbf{y}) = \sum_{j_1, j_2 = 1}^{n} \frac{\partial^2 \mathbf{f}(\mathbf{x}_0)}{\partial x_{j_1} \partial x_{j_2}} x_{j_1} y_{j_2}.$$

Similar formulas hold for $D^k\mathbf{f}(\mathbf{x}_0) : (E \times \cdots \times E) \to \mathbf{R}^n$.

A function $\mathbf{f} : E \to \mathbf{R}^n$ is said to be analytic in the open set $E \subset \mathbf{R}^n$ if each component $f_j(\mathbf{x}), j = 1, \ldots, n$, is analytic in E, i.e., if for $j = 1, \ldots, n$ and $\mathbf{x}_0 \in E$, $f_j(\mathbf{x})$ has a Taylor series which converges to $f_j(\mathbf{x})$ in some neighborhood of $\mathbf{x}_0$ in E.

1.2 The Fundamental Existence-Uniqueness Theorem

In this section, we establish the fundamental existence-uniqueness theorem for a nonlinear autonomous system of ordinary differential equations (2) under the hypothesis that $\mathbf{f} \in C^1(E)$ where E is an open subset of $\mathbf{R^n}$. Picard's classical method of successive approximations is used to prove this theorem. The more modern approach based on the contraction mapping principle is relegated to the problems at the end of this section. The method of successive approximations not only allows us to establish the existence and uniqueness of the solution of the initial value problem associated with (2), but it also allows us to establish the continuity and differentiability of the solution with respect to initial conditions and parameters. This is done in the next section. The method is also used in the proof of the Stable Manifold and in the proof of the Hartman-Grobman. The method of successive approximations is one of the basic tools used in the qualitative theory of ordinary differential equations.

Definition 1. Suppose that $\mathbf{f} \in C(E)$ where E is an open subset of $\mathbf{R}^n$. Then $\mathbf{x}(t)$ is a solution of the differential equation (2) on an interval I if $\mathbf{x}(t)$ is differentiable on I and if for all $t \in I, \mathbf{x}(t) \in E$ and

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$$

And given $x_0 \in E, x(t)$ is a solution of the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$
$$\mathbf{x}(t_0) = \mathbf{x}_0$$

on an interval I if $t_0 \in I, \mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t)$ is a solution of the differential equation (2) on the interval I.

In order to apply the method of successive approximations to establish the existence of a solution of (2), we need to define the concept of a Lipschitz condition and show that C^1 functions are locally Lipschitz.

Definition 2. Let E be an open subset of $\mathbf{R}^n$. A function $\mathrm{f} : E \to \mathbf{R}^n$ is said to satisfy a Lipschitz condition on E if there is a positive constant K such that for all $x, y \in E$

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|.$$

The function $\mathbf{f}$ is said to be locally Lipschitz on E if for each point $\mathbf{x}_0 \in E$ there is an ε-neighborhood of $\mathrm{x}_0, N_\varepsilon(\mathrm{x}_0) \subset E$ and a constant $K_0 > 0$ such that for all $\mathbf{x}, \mathbf{y} \in N_\varepsilon(x_0)$

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K_0|\mathbf{x} - \mathbf{y}|.$$

By an ε-neighborhood of a point $\mathbf{x}_0 \in \mathbf{R}^n$, we mean an open ball of positive radius ε; i.e.,

$$N_\varepsilon(x_0) = \{\mathbf{x} \in \mathbf{R}^n \| \mathbf{x} - \mathrm{x}_0 | < \varepsilon\}.$$

Lemma. Let E be an open subset of $\mathbf{R^n}$ and let $\mathbf{f} : E \to \mathbf{R}^n$. Then, if $\mathbf{f} \in C^1(E)$, $\mathbf{f}$ is locally Lipschitz on E.

Proof. Since E is an open subset of $\boldsymbol{R^n}$, given $\mathrm{x}_0 \in E$, there is an $\varepsilon > 0$ such that $N_\varepsilon(\mathrm{x}_0) \subset E$. Let

$$K = \max_{|\mathrm{x}-\mathrm{x}_0| \leq \varepsilon/2} \|D\mathbf{f}(\mathrm{x})\|,$$

the maximum of the continuous function $D\mathbf{f}(\mathbf{x})$ on the compact set $|\ x - \mathrm{x}_0 \ | \leq \varepsilon/2$. Let N_0 denote the $\varepsilon/2$-neighborhood of $\mathrm{x}_0, N_{\varepsilon/2}(\mathrm{x}_0)$. Then for $\mathbf{x}, \mathbf{y} \in N_0$, set $\mathbf{u} = \mathbf{y} - \mathbf{x}$. It follows that $\mathbf{x} + s\mathbf{u} \in N_0$ for $0 \leq s \leq 1$ since N_0 is a convex set. Define the function $F : [0, 1] \to \mathbf{R}^n$ by

$$\mathbf{F}(s) = \mathbf{f}(\mathbf{x} + s\mathbf{u}).$$

Then by the chain rule,

$$\mathbf{F}'(s) = D\mathbf{f}(\mathbf{x} + s\mathbf{u})\mathbf{u}$$

and therefore

$$\begin{aligned} \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) &= \mathbf{F}(1) - \mathbf{F}(0) \\ &= \int_0^1 \mathbf{F}'(s)ds = \int_0^1 D\mathbf{f}(\mathbf{x} + s\mathbf{u})\mathbf{u}ds. \end{aligned}$$

It then follows from the lemma that

$$\begin{aligned} |\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| &\leq \int_0^1 |D\mathbf{f}(\mathbf{x} + s\mathrm{u})\mathrm{u}| ds \\ &\leq \int_0^1 \|D\mathbf{f}(\mathbf{x} + s\mathrm{u})\| |\mathbf{u}| ds \\ &\leq K|\mathbf{u}| = K|\mathbf{y} - \mathbf{x}|. \end{aligned}$$

And this proves the lemma. Picard's method of successive approximations is based on the fact that $\mathbf{x}(t)$ is a solution of the initial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned} \tag{4}$$

if and only if $\mathbf{x}(t)$ is a continuous function that satisfies the integral equation

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{f}(\mathbf{x}(s))ds.$$

The successive approximations to the solution of this integral equation are defined by the sequence of functions

$$\begin{aligned} \mathbf{u}_0(t) &= \mathbf{x}_0 \\ \mathbf{u}_{k+1}(t) &= \mathbf{x}_0 + \int_0^t \mathbf{f}\left(\mathbf{u}_k(s)\right) ds \end{aligned} \tag{5}$$

for $k = 0, 1, 2, \ldots$. In order to illustrate the mechanics involved in the method of successive approximations, we use the method to solve an elementary linear differential equation.

Definition 3. Let V be a normed linear space. Then a sequence $\{u_k\} \subset V$ is called a Cauchy sequence if for all $\varepsilon > 0$ there is an N such that $k, m \geq N$ implies that

$$||u_k - \mathbf{u}_m|| < \varepsilon.$$

The space V is called complete if every Cauchy sequence in V converges to an element in V.

The following theorem, establishes the completeness of the normed linear space $C(I)$ with $I = [-a, a]$.

Theorem. For $I = [-a, a], C(I)$ is a complete normed linear space.
We can now prove the fundamental existence-uniqueness theorem for nonlinear systems.

Theorem (The Fundamental Existence-Uniqueness Theorem). Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$ and assume that $\mathbf{f} \in C^1(E)$. Then there exists an $a > 0$ such that the initial value problem (4) has a unique solution $\mathbf{x}(t)$ on the interval $[-a, a]$.

Proof. Since f $\in C^1(E)$, it follows from the lemma that there is an ε neighborhood $N_\varepsilon\left(\mathrm{x}_0\right) \subset E$ and a constant $K > 0$ such that for all $\mathbf{x}, \mathbf{y} \in N_\varepsilon\left(\mathbf{x}_0\right)$,

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|.$$

Let $b = \varepsilon/2$. Then the continuous function $\mathbf{f}(\mathbf{x})$ is bounded on the compact set

$$N_0 = \{\mathbf{x} \in \mathbf{R}^n ||\mathbf{x} - \mathbf{x}_0 | \leq b\}.$$

Let

$$M = \max_{\mathbf{x} \in N_0} |\mathbf{f}(\mathbf{x})|.$$

Let the successive approximations $u_k(t)$ be defined by (5). Then assuming that there exists an $a > 0$ such that $u_k(t)$ is defined and continuous on $[-a, a]$ and satisfies

$$\max_{[-a,a]} |u_k(t) - x_0| \leq b, \tag{6}$$

it follows that $\mathrm{f}(\mathrm{u}_k(t))$ is defined and continuous on $[-a, a]$ and therefore that

$$\mathbf{u}_{k+1}(t) = \mathrm{x}_0 + \int_0^t \mathrm{f}(\mathbf{u}_k(s))\, ds$$

is defined and continuous on $[-a, a]$ and satisfies

$$|\mathrm{u}_{k+1}(t) - \mathrm{x}_0| \leq \int_0^t |\mathbf{f}(\mathrm{u}_k(s))|\, ds \leq Ma$$

for all $t \in [-a, a]$. Thus, choosing $0 < a \leq b/M$, it follows by induction that $u_k(t)$ is defined and continuous and satisfies (6) for all $t \in [-a, a]$ and $k = 1, 2, 3, \ldots$.

Next, since for all $t \in [-a, a]$ and $k = 0, 1, 2, 3, \ldots, \mathbf{u}_k(t) \in N_0$, it follows from the Lipschitz condition satisfied by f that for all $t \in [-a, a]$

$$\begin{aligned}
|\mathbf{u}_2(t) - \mathbf{u}_1(t)| &\leq \int_0^t |\mathbf{f}(\mathbf{u}_1(s)) - \mathbf{f}(\mathbf{u}_0(s))|\, ds \\
&\leq K \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_0(s)|\, ds \\
&\leq Ka \max_{[-a,a]} |\mathbf{u}_1(t) - \mathbf{x}_0| \\
&\leq Kab.
\end{aligned}$$

And then assuming that

$$\max_{[-a,a]} \left|\mathbf{u}_j(t) - \mathbf{u}_{j-1}(t)\right| \leq (Ka)^{j-1} b \tag{7}$$

for some integer $j \geq 2$, it follows that for all $t \in [-a, a]$

$$\begin{aligned}
\left|\mathbf{u}_{j+1}(t) - \mathbf{u}_j(t)\right| &\leq \int_0^t \left|\mathbf{f}\left(\mathbf{u}_j(s)\right) - \mathbf{f}\left(\mathbf{u}_{j-1}(s)\right)\right| ds \\
&\leq K \int_0^t \left|\mathbf{u}_j(s) - \mathbf{u}_{j-1}(s)\right| ds \\
&\leq Ka \max_{[-a,a|} \left|\mathbf{u}_j(t) - \mathbf{u}_{j-1}(t)\right| \\
&\leq (Ka)^j b.
\end{aligned}$$

Thus, it follows by induction that (7) holds for $j = 2, 3, \ldots$. Setting $\alpha =$ and choosing $0 < a < 1/K$, we see that for $m > k \geq N$ and $t \in [-a, a$

$$\begin{aligned}|\mathbf{u}_m(t) - \mathbf{u}_k(t)| &\leq \sum_{j=k}^{m-1} \left|\mathbf{u}_{j+1}(t) - \mathbf{u}_j(t)\right| \\ &\leq \sum_{j=N}^{\infty} \left|\mathbf{u}_{j+1}(t) - \mathbf{u}_j(t)\right| \\ &\leq \sum_{j=N}^{\infty} \alpha^j b = \frac{\alpha^N}{1-\alpha} b.\end{aligned}$$

This last quantity approaches zero as $N \to \infty$. Therefore, for all ε there exists an N such that $m, k \geq N$ implies that

$$\|\mathbf{u}_m - \mathbf{u}_k\| = \max_{|-a,a|} |\mathbf{u}_m(t) - \mathbf{u}_k(t)| < \varepsilon;$$

i.e., $\{u_k\}$ is a Cauchy sequence of continuous functions in $C([-a, a])$. I lows from the above theorem that $u_k(t)$ converges to a continuous funs $\mathbf{u}(t)$ uniformly for all $t \in [-a, a]$ as $k \to \infty$. And then taking the lim both sides of equation (3) defining the successive approximations, w that the continuous function

$$\mathbf{u}(t) = \lim_{k\to\infty} \mathbf{u}_k(t) \tag{8}$$

satisfies the integral equation

$$\mathbf{u}(t) = \mathbf{x}_0 + \int_0^t \mathbf{f}(\mathbf{u}(s))ds \tag{9}$$

for all $t \in [-a, a]$. We have used the fact that the integral and the limit can be interchanged since the limit in (8) is uniform for all $t \in [-a, a]$. Then since $\mathbf{u}(t)$ is continuous, $\mathbf{f}(\mathbf{u}(t))$ is continuous and by the fundamental theorem of calculus, the right-hand side of the integral equation (9) is differentiable and

$$\mathbf{u}'(t) = \mathbf{f}(\mathbf{u}(t))$$

for all $t \in [-a, a]$. Furthermore, $\mathbf{u}(0) = \mathbf{x}_0$ and from (6) it follows that $\mathbf{u}(t) \in N_c(\mathbf{x}_0) \subset E$ for all $t \in [-a, a]$. Thus $\mathbf{u}(t)$ is a solution of the initial value problem (4) on $[-a, a]$. It remains to show that it is the only solution.

Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be two solutions of the initial value problem (4) on $[-a, a]$. Then the continuous function $|\mathbf{u}(t) - \mathbf{v}(t)|$ achieves its maximum at some point $t_1 \in [-a, a]$. It follows that

$$\begin{aligned}
\|\mathbf{u} - \mathbf{v}\| &= \max_{[-a,a]} |\mathbf{u}(t) - \mathbf{v}(t)| \\
&= \left| \int_0^{t_1} \mathbf{f}(\mathbf{u}(s)) - \mathbf{f}(\mathbf{v}(s)) ds \right| \\
&\leq \int_0^{|t_1|} |\mathbf{f}(\mathbf{u}(s)) - \mathbf{f}(\mathbf{v}(s))| ds \\
&\leq K \int_0^{|t_1|} |\mathbf{u}(s) - \mathbf{v}(s)| ds \\
&\leq Ka \max |\mathbf{u}(t) - \mathbf{v}(t)| \\
&\leq Ka \|\mathbf{u} - \mathbf{v}\|
\end{aligned}$$

But $Ka < 1$ and this last inequality can only be satisfied if $\|\mathbf{u} - \mathbf{v}\| = 0$. Thus, $\mathbf{u}(t) = \mathbf{v}(t)$ on $[-a, a]$. We have shown that the successive approximations (5) converge uniformly to a unique solution of the initial value problem (4) on the interval $[-a, a]$ where a is any number satisfying $0 < a < \min\left(\frac{b}{M}, \frac{1}{K}\right)$

Remark. Exactly the same method of proof shows that the initial value problem

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\
\mathbf{x}(t_0) &= \mathbf{x}_0
\end{aligned}$$

has a unique solution on some interval $[t_0 - a, t_0 + a]$.

1.3 Dependence on Initial Conditions and Parameters

In this section we investigate the dependence of the solution of the initial value problem

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\
\mathbf{x}(0) &= \mathbf{y}
\end{aligned} \tag{1}$$

on the initial condition $\mathbf{y}$. If the differential equation depends on a parameter $\mu \in R^m$, i.e., if the function $f(\mathbf{x})$ in (1) is replaced by $f(\mathbf{x}, \mu)$, then the solution $\mathbf{u}(t, \mathbf{y}, \boldsymbol{\mu})$ will also depend on the parameter $\boldsymbol{\mu}$. Roughly speaking, the dependence of the solution $\mathbf{u}(t, \mathbf{y}, \boldsymbol{\mu})$ on the initial condition $\mathbf{y}$ and the parameter $\boldsymbol{\mu}$ is as continuous as the function f. In order to establish this type of continuous dependence of the solution on initial conditions and parameters, we first establish a result due to T.H. Gronwall.

Lemma (Gronwall). Suppose that $g(t)$ is a continuous real valued function that satisfies $g(t) \geq 0$ and

$$g(t) \leq C + K \int_0^t g(s)ds$$

for all $t \in [0, a]$ where C and K are positive constants. It then follows that for all $t \in [0, a]$,

$$g(t) \leq Ce^{Kt}$$

Proof

Let $G(t) = C + K \int_0^t g(s)ds$ for $t \in [0, a]$. Then $G(t) \geq g(t)$ and $G(t) > 0$ for all $t \in [0, a]$. It follows from the fundamental theorem of calculus that

$$G'(t) = Kg(t)$$

and therefore that

$$\frac{G'(t)}{G(t)} = \frac{Kg(t)}{G(t)} \leq \frac{KG(t)}{G(t)} = K$$

for all $t \in [0, a]$. And this is equivalent to saying that

$$\frac{d}{dt}(\log G(t)) \leq K$$

or

$$\log G(t) \leq Kt + \log G(0)$$

or

$$G(t) \leq G(0)e^{Kt} = Ce^{Kt}$$

for all $t \in [0, a]$, which implies that $g(t) \leq Ce^{Kt}$ for all $t \in [0, a]$.

Theorem (Dependence on Initial Conditions).

Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$ and assume that $\mathbf{f} \in C^1(E)$. Then there exists an $a > 0$ and $a\delta > 0$ such that for all $\mathbf{y} \in N_\delta(\mathbf{x}_0)$ the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$$\mathbf{x}(0) = \mathbf{y}$$

has a unique solution $\mathbf{u}(t, \mathbf{y})$ with $\mathbf{u} \in C^1(G)$ where $G = [-a, a] \times N_6(\mathbf{x}_0) \subset \mathbf{R}^{n+1}$; furthermore, for each

$\mathbf{y} \in N_\delta(\mathbf{x}_0)$, $\mathbf{u}(t, \mathbf{y})$ is a twice continuously differentiable function of t for $t \in [-a, a]$.

Proof.

Since f $\in C^1(E)$, it follows from the lemma in Section 2.2 that there is an ε-neighborhood $N_\varepsilon(\mathbf{x}_0) \subset E$ and a constant $K > 0$ such that for all $\mathbf{x}$ and $\mathbf{y} \in N_\varepsilon(\mathbf{x}_0)$,

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|$$

As in the proof of the fundamental existence theorem, let $N_0 = \{\mathbf{x} \in \mathbf{R}^n \mid |\mathbf{x} - \mathbf{x}_0| \leq \varepsilon/2\}$, let M_0 be the maximum of $|\mathbf{f}(\mathbf{x})|$ on N_0 and let M_1 be the maximum of $\|D\mathbf{f}(\mathbf{x})\|$ on N_0. Let $\delta = \varepsilon/4$, and for $\mathbf{y} \in N_\delta(\mathbf{x}_0)$ define the successive approximations $\mathbf{u}_k(t, \mathbf{y})$ as

$$\begin{aligned} \mathbf{u}_0(t, \mathbf{y}) &= \mathbf{y} \\ \mathbf{u}_{k+1}(t, \mathbf{y}) &= \mathbf{y} + \int_0^t \mathbf{f}(\mathbf{u}_k(s, \mathbf{y}))\, ds \end{aligned} \tag{2}$$

Assume that $\mathbf{u}_k(t, \mathbf{y})$ is defined and continuous for all $(t, \mathbf{y}) \in G = [-a, a] \times N_\delta(\mathbf{x}_0)$ and that for all $\mathbf{y} \in N_\delta(\mathbf{x}_0)$

$$\left\|\mathbf{u}_k(t, \mathbf{y}) - \mathbf{x}_0\right\| < \varepsilon/2 \tag{3}$$

where $\|\cdot\|$ denotes the maximum over all $t \in [-a, a]$. This is clearly satisfied for $k = 0$. And assuming this is true for k, it follows that $\mathbf{u}_{k+1}(t, \mathbf{y})$, defined by the above successive approximations, is continuous on G. This follows since a continuous function of a continuous function is continuous and since the above integral of the continuous function f$(\mathbf{u}_k(s, y))$ is continuous in t by the fundamental theorem of calculus and also in $\mathbf{y}$; cf. Rudin [R] or Carslaw [C]. We also have

$$\left\|\mathbf{u}_{k+1}(t, \mathbf{y}) - \mathbf{y}\right\| \leq \int_0^t \left|\mathbf{f}(\mathbf{u}_k(s, \mathbf{y}))\right| ds \leq M_0 a$$

for $t \in [-a, a]$ and $\mathbf{y} \in N_\delta(\mathbf{x}_0) \subset N_0$. Thus, for $t \in [-a, a]$ and $\mathbf{y} \in N_\delta(\mathbf{x}_0)$ with $\delta = \varepsilon/4$, we have

$$\begin{aligned} \left\|\mathbf{u}_{k+1}(t, \mathbf{y}) - \mathbf{x}_0\right\| &\leq \left\|\mathbf{u}_{k+1}(t, \mathbf{y}) - \mathbf{y}\right\| + \left\|\mathbf{y} - \mathbf{x}_0\right\| \\ &\leq M_0 a + \varepsilon/4 < \varepsilon/2 \end{aligned}$$

provided $M_0 a < \varepsilon/4$, i.e., provided $a < \varepsilon/(4M_0)$. Thus, the above induction hypothesis holds for all $k = 1, 2, 3, \ldots$ and $(t, y) \in G$ provided $a < \varepsilon/(4M_0)$.

We next show that the successive approximations $\mathbf{u}_k(t, y)$ converge uniformly to a continuous function $\mathbf{u}(t, \mathbf{y})$ for all $(t, \mathbf{y}) \in G$ as $k \to \infty$. As in the proof of the fundamental existence theorem,

$$\begin{aligned}\left\|\mathbf{u}_2(t,\mathbf{y})-\mathbf{u}_1(t,\mathbf{y})\right\| &\le Ka\left\|\mathbf{u}_1(t,\mathbf{y})-\mathbf{y}\right\|\\ &\le Ka\left\|\mathbf{u}_1(t,\mathbf{y})-\mathbf{x}_0\right\|+Ka\left\|\mathbf{y}-\mathbf{x}_0\right\|\\ &\le Ka(\varepsilon/2+\varepsilon/4)\le Ka\varepsilon\end{aligned}$$

for $(t,\mathbf{y})\in G$. And then it follows exactly as in the proof of the fundamental existence theorem in Section 2.2 that

$$\left\|\mathbf{u}_{k+1}(t,\mathbf{y})-\mathbf{u}_k(t,\mathbf{y})\right\|\le (Ka)^k\varepsilon$$

for $(t,\mathbf{y})\in G$ and consequently that the successive approximations converge uniformly to a continuous function $\mathbf{u}(t,\mathbf{y})$ for $(t,\mathbf{y})\in G$ as $k\to\infty$ provided $a<1/K$. Furthermore, the function $\mathbf{u}(t,\mathbf{y})$ satisfies

$$\mathbf{u}(t,\mathbf{y})=\mathbf{y}+\int_0^t \mathbf{f}(\mathbf{u}(s,\mathbf{y}))ds$$

for $(t,\mathbf{y})\in G$ and also $\mathbf{u}(0,\mathbf{y})=\mathbf{y}$. And it follows from the inequality (3) that $\mathbf{u}(t,\mathbf{y})\in N_{\varepsilon/2}(\mathbf{x}_0)$ for all $(t,\mathbf{y})\in G$. Thus, by the fundamental theorem of calculus and the chain rule, it follows that

$$\dot{\mathbf{u}}(t,\mathbf{y})=\mathbf{f}(\mathbf{u}(t,\mathbf{y}))$$

and that

$$\ddot{\mathbf{u}}(t,\mathbf{y})=D\mathbf{f}(\mathbf{u}(t,\mathbf{y}))\dot{\mathbf{u}}(t,\mathbf{y})$$

for all $(t,y)\in G$; i.e., $\mathbf{u}(t,y)$ is a twice continuously differentiable function of t which satisfies the initial value problem (1) for all $(t,y)\in G$. The uniqueness of the solution $\mathbf{u}(t,y)$ follows from the fundamental theorem in Section 2.2.

We now show that $\mathbf{u}(t,y)$ is a continuously differentiable function of $\mathbf{y}$ for all $(t,y)\in[-a,a]\times N_{\delta/2}(x_0)$. In order to do this, fix $\mathbf{y}_0\in N_{\delta/2}(\mathbf{x}_0)$ and choose $\mathbf{h}\in\mathbf{R}^\mathbf{n}$ such that $|\mathbf{h}|<\delta/2$. Then $\mathbf{y}_0+\mathbf{h}\in N_\delta(\mathbf{x}_0)$. Let $\mathbf{u}(t,\mathbf{y}_0)$ and $\mathbf{u}(t,\mathbf{y}_0+\mathbf{h})$ be the solutions of the initial value problem (1) with $\mathbf{y}=\mathbf{y}_0$ and with $y=y_0+\mathbf{h}$ respectively. It then follows that

$$\begin{aligned}\left|\mathbf{u}(t,\mathbf{y}_0+\mathbf{h})-\mathbf{u}(t,\mathbf{y}_0)\right| &\le |\mathbf{h}|+\int_0^t\left|\mathbf{f}(\mathbf{u}(s,\mathbf{y}_0+\mathbf{h}))-\mathbf{f}(\mathbf{u}(s,\mathbf{y}_0))\right|ds\\ &\le |\mathbf{h}|+K\int_0^t\left|\mathbf{u}(s,\mathbf{y}_0+\mathbf{h})-\mathbf{u}(s,\mathbf{y}_0)\right|ds\end{aligned}$$

for all $t\in[-a,a]$. Thus, it follows from Gronwall's Lemma that

$$\left|\mathbf{u}(t,\mathbf{y}_0+\mathbf{h})-\mathbf{u}(t,\mathbf{y}_0)\right|\le|\mathbf{h}|e^{K|t|} \tag{4}$$

for all $t\in[-a,a]$. We next define $\Phi(t,y_0)$ to be the fundamental matrix solution of the initial value problem

$$\dot{\Phi} = A(t, \mathbf{y}_0)\Phi$$
$$\Phi(0, \mathbf{y}_0) = I \tag{5}$$

with $A(t, y_0) = D\mathbf{f}(\mathbf{u}(t, \mathbf{y}_0))$ and I the $n \times n$ identity matrix. The existence and continuity of $\Phi(t, y_0)$ on some interval $[-a, a]$ follow from the method of successive approximations as in problem 4 of Problem Set 2 and problem 4 in Problem Set 3. It then follows from the initial value problems for $\mathbf{u}(t, y_0)$, $\mathbf{u}(t, \mathbf{y}_0 + \mathbf{h})$ and $\Phi(t, \mathbf{y}_0)$ and Taylor's Theorem,

$$\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_0) = D\mathbf{f}(\mathbf{u}_0)(\mathbf{u} - \mathbf{u}_0) + \mathbf{R}(\mathbf{u}, \mathbf{u}_0)$$

where $|\mathbf{R}(\mathbf{u}, \mathbf{u}_0)| / |\mathbf{u} - \mathbf{u}_0| \to 0$ as $|\mathbf{u} - \mathbf{u}_0| \to 0$, that

$$\begin{aligned}
|\, \mathbf{u}(t, \mathbf{y}_0) - \mathbf{u}(t, \mathbf{y}_0 + \mathbf{h}) + \Phi(t, \mathbf{y}_0)\mathbf{h} \Big| &\leq \int_0^t \Big| \mathbf{f}(\mathbf{u}(s, \mathbf{y}_0)) \\
&- \mathbf{f}(\mathbf{u}(s, \mathbf{y}_0 + \mathbf{h})) + D\mathbf{f}(\mathbf{u}(s, \mathbf{y}_0))\Phi(s, \mathbf{y}_0)\mathbf{h} \,|\, ds \\
&\leq \int_0^t \left\| D\mathbf{f}(\mathbf{u}(s, \mathbf{y}_0)) \right\| \left| \mathbf{u}(s, \mathbf{y}_0) - \mathbf{u}(s, \mathbf{y}_0 + \mathbf{h}) + \Phi(s, \mathbf{y}_0)\mathbf{h} \right| ds \\
&+ \int_0^t \left| \mathbf{R}(\mathbf{u}(s, \mathbf{y}_0 + \mathbf{h}), \mathbf{u}(s, \mathbf{y}_0)) \right| ds
\end{aligned} \tag{6}$$

Since $|\mathbf{R}(u, u_0)| / |u - u_0| \to 0$ as $|\mathbf{u} - \mathbf{u}_0| \to 0$ and since $\mathbf{u}(s, \mathbf{y})$ is continuous on G, it follows that given any $\varepsilon_0 > 0$, there exists a $\delta_0 > 0$ such that if $|\mathbf{h}| < \delta_0$ then $\left|\mathbf{R}(\mathbf{u}(s, \mathbf{y}_0), \mathbf{u}(s, \mathbf{y}_0 + \mathbf{h}))\right| < \varepsilon_0 \left|\mathbf{u}(s, \mathbf{y}_0) - \mathbf{u}(s, \mathbf{y}_0 + \mathbf{h})\right|$ for all $s \in [-a, a]$. Thus, if we let

$$g(t) = \left|\mathbf{u}(t, \mathbf{y}_0) - \mathbf{u}(t, \mathbf{y}_0 + \mathbf{h}) + \Phi(t, \mathbf{y}_0)\mathbf{h}\right|$$

it then follows from (4) and (6) that for all $t \in [-a, a]$, $y_0 \in N_{\delta/2}(x_0)$ and $|\mathbf{h}| < \min(\delta_0, \delta/2)$ we have

$$g(t) \leq M_1 \int_0^t g(s)ds + \varepsilon_0 |\mathbf{h}| a e^{Ka}.$$

Hence, it follows from Gronwall's Lemma that for any given $\varepsilon_0 > 0$

$$g(t) \leq \varepsilon_0 |\mathbf{h}| a \mathrm{e}^{Ka} e^{M_1 \mathrm{a}}$$

for all $t \in [-a, a]$ provided $|\mathbf{h}| < \min(\delta_0, \delta/2)$. Thus,

$$\lim_{|\mathbf{h}| \to 0} \frac{\left|\mathbf{u}(t, \mathbf{y}_0) - \mathbf{u}(t, \mathbf{y}_0 + \mathbf{h}) + \Phi(t, \mathbf{y}_0)\,\mathbf{h}\right|}{|\mathbf{h}|} = 0$$

uniformly for all $t \in [-a, a]$. Therefore, according to Definition 1 in Section 2.1,

$$\frac{\partial \mathbf{u}}{\partial \mathbf{y}}(t, \mathbf{y}_0) = \Phi(t, \mathbf{y}_0)$$

for all $t \in [-a, a]$ where $\Phi(t, y_0)$ is the fundamental matrix solution of the initial value problem (5) which is continuous in t and in y_0 for all $t \in [-a, a]$ and $y_0 \in N_{\delta/2}(x_0)$. This completes the proof of the theorem.

Corollary. Under the hypothesis of the above theorem,

$$\Phi(t, \mathbf{y}) = \frac{\partial \mathbf{u}}{\partial \mathbf{y}}(t, \mathbf{y})$$

for $t \in [-a, a]$ and $\mathbf{y} \in N_\delta(\mathrm{x}_0)$ if and only if $\Phi(t, y)$ is the fundamental matrix solution of

$$\Phi = D\mathbf{f}[\mathbf{u}(t, \mathbf{y})]\Phi$$
$$\Phi(0, \mathbf{y}) = I$$

for $t \in [-a, a]$ and $\mathrm{y} \in N_\delta(\mathrm{x}_0)$.

Remark 1. A similar proof shows that if $f \in C^r(E)$ then the solution $\mathbf{u}(t, \mathbf{y})$ of the initial value problem (1) is in $C^r(G)$ where G is defined as in the above theorem. And if $\mathbf{f}(\mathbf{x})$ is a (real) analytic function for $\mathbf{x} \in E$ then $\mathbf{u}(t, \mathbf{y})$ is analytic in the interior of G; cf. [C/L].

Remark 2. If x_0 is an equilibrium point of (1), i.e., if $f(x_0) = 0$ so that $\mathbf{u}(t, \mathbf{x}_0) = \mathbf{x}_0$ for all $t \in \mathbf{R}$, then

$$\Phi(t, \mathbf{x}_0) = \frac{\partial \mathrm{u}}{\partial \mathbf{x}_0}(t, \mathbf{x}_0)$$

satisfies

$$\dot{\Phi} = D\mathbf{f}(\mathbf{x}_0)\,\Phi$$
$$\Phi(0, \mathbf{x}_0) = I.$$

And according to the Fundamental Theorem for Linear Systems

$$\Phi(t, \mathrm{x}_0) = e^{D\mathbf{f}(\mathrm{x}_0)t}.$$

Remark 3. It follows from the continuity of the solution $\mathbf{u}(t, \mathbf{y})$ of the initial value problem (1) that for each $t \in [-a, a]$

$$\lim_{\mathbf{y}\to\mathbf{x}_0} \mathbf{u}(t,\mathbf{y}) = \mathbf{u}\left(t,\mathbf{x}_0\right).$$

It follows from the inequality (4) that this limit is uniform for all $t \in [-a, a]$. We prove a slightly stronger version of this result in Theorem 4 of the next section.

Theorem 2 (Dependence on Parameters). Let E be an open subset of $\mathbf{R}^{n+m}$ containing the point $(\mathbf{x}_0, \mu_0)$ where $\mathbf{x}_0 \in \mathbf{R}^n$ and $\mu_0 \in \mathbf{R}^m$ and assume that $\mathbf{f} \in C^1(E)$. It then follows that there exists an $a > 0$ and a $\delta > 0$ such that for all $\mathbf{y} \in N_\delta(\mathbf{x}_0)$ and $\mu \in N_\delta(\mu_0)$, the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\mu})$$

$$\mathbf{x}(0) = \mathbf{y}$$

has a unique solution $\mathbf{u}(t, \mathbf{y}, \boldsymbol{\mu})$ with $\mathbf{u} \in C^1(G)$ where $G = [-a, a] \times N_\delta(\mathbf{x}_0) \times N_\delta(\mu_0)$.

This theorem follows immediately from the previous theorem by replacing the vectors $\mathbf{x}_0, \mathbf{x}, \dot{\mathbf{x}}$ and $\mathbf{y}$ by the vectors $(\mathbf{x}_0, \mu_0), (\mathbf{x}, \mu), (\dot{\mathbf{x}}, \mathbf{0})$ and $(\mathbf{y}, \boldsymbol{\mu})$ or it can be proved directly using Gronwall's Lemma and the method of successive approximations.

Lemma 1. Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$ and suppose $\mathbf{f} \in C^1(E)$. Let $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ be solutions of the initial value problem (1) on the intervals I_1 and I_2. Then $0 \in I_1 \cap I_2$ and if I is any open interval containing 0 and contained in $I_1 \cap I_2$, it follows that $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ for all $t \in I$. Proof. Since $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are solutions of the initial value problem (1) on I_1 and I_2 respectively, it follows from Definition 1 in Section 2.2 that $0 \in I_1 \cap I_2$. And if I is an open interval containing 0 and contained in $I_1 \cap I_2$, then the fundamental existence-uniqueness theorem in Section 2.2 implies that $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ on some open interval $(-a, a) \subset I$. Let I^* be the union of all such open intervals contained in I. Then I^* is the largest open interval contained in I on which $\mathbf{u}_1(t) = \mathbf{u}_2(t)$. Clearly, $I^* \subset I$ and if I^* is a proper subset of I, then one of the endpoints t_0 of I^* is contained in $I \subset I_1 \cap I_2$. It follows from the continuity of $u_1(t)$ and $u_2(t)$ on I that

$$\lim_{t\to t_0} \mathbf{u}_1(t) = \lim_{t\to t_0} \mathbf{u}_2(t).$$

Call this common limit $\mathbf{u}_0$. It then follows from the uniqueness of solutions that $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ on some interval $I_0 = (t_0 - a, t_0 + a) \subset I$. Thus, $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ on the interval $I^* \cup I_0 \subset I$ and I^* is a proper subset of $I^* \cup I_0$. But this contradicts the fact that I^* is the largest open interval contained in I on which $\mathbf{u}_1(t) = \mathbf{u}_2(t)$. Therefore, $I^* = I$ and we have $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ for all $t \in I$.

Theorem 1. Let E be an open subset of $\mathbf{R}^n$ and assume that $\mathbf{f} \in C^1(E)$. Then for each point $x_0 \in E$, there is a maximal interval J on which the initial value problem (1) has a unique solution, $\mathbf{x}(t)$; i.e., if the initial value problem has a solution $\mathbf{y}(t)$ on an interval I then $I \subset J$ and $\mathbf{y}(t) = \mathbf{x}(t)$ for all $t \in I$. Furthermore, the maximal interval J is open; i.e., $J = (\alpha, \beta)$.

Proof. By the fundamental existence-uniqueness theorem in Section 2.2, the initial value problem (1) has a unique solution on some open interval $(-a, a)$. Let (α, β) be the union of all open intervals I such that (1) has a solution on I. We define a function $\mathbf{x}(t)$ on (α, β) as follows: Given $t \in (\alpha, \beta)$, t belongs to some open interval I such that (1) has a solution $\mathbf{u}(t)$ on I; for this given $t \in (\alpha, \beta)$, define $\mathbf{x}(t) = \mathbf{u}(t)$. Then $\mathbf{x}(t)$ is a well-defined function of t since if $t \in I_1 \cap I_2$ where I_1 and I_2 are any two open intervals such that (1) has solutions $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ on I_1 and I_2 respectively, then by the lemma $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ on the open interval $I_1 \cap I_2$. Also, $\mathbf{x}(t)$ is a solution of (1) on (α, β) since each point $t \in (\alpha, \beta)$ is contained in some open interval I on which the initial value problem (1) has a unique solution $\mathbf{u}(t)$ and since $\mathbf{x}(t)$ agrees with $\mathbf{u}(t)$ on I. The fact that J is open follows from the fact that any solution of (1) on an interval $(\alpha, \beta]$ can be uniquely continued to a solution on an interval $(\alpha, \beta + a)$ with $a > 0$ as in the proof of Theorem 2 below.

Definition. The interval (α, β) in Theorem 1 is called the maximal interval of existence of the solution $\mathbf{x}(t)$ of the initial value problem (1) or simply the maximal interval of existence of the initial value problem (1).

Theorem 2. Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$, let $\mathbf{f} \in C^1(E)$, and let (α, β) be the maximal interval of existence of the solution $\mathbf{x}(t)$ of the initial value problem (1). Assume that $\beta < \infty$. Then given any compact set $K \subset E$, there exists a $t \in (\alpha, \beta)$ such that $\mathbf{x}(t) \notin K$.

Proof. Since f is continuous on the compact set K, there is a positive number M such that $|\mathbf{f}(\mathbf{x})| \le M$ for all $\mathbf{x} \in K$. Let $\mathbf{x}(t)$ be the solution of the initial value problem (1) on its maximal interval of existence (α, β) and assume that $\beta < \infty$ and that $\mathbf{x}(t) \in K$ for all $t \in (\alpha, \beta)$. We first show that $\lim_{t \to \beta^-} x(t)$ exists. If $\alpha < t_1 < t_2 < \beta$ then

$$|\mathbf{x}(t_1) - \mathbf{x}(t_2)| \le \int_{t_1}^{t_2} |\mathbf{f}(\mathbf{x}(s))| ds \le M |t_2 - t_1| .$$

Thus as t_1 and t_2 approach β from the left, $|\mathbf{x}(t_2) - \mathbf{x}(t_1)| \to 0$ which, by the Cauchy criterion for convergence in $\mathbf{R}^n$ (i.e., the completeness of $\mathbf{R}^n$) implies that $\lim_{t \to \beta^-} \mathbf{x}(t)$ exists. Let $\mathbf{x}_1 = \lim_{t \to \beta^-} \mathbf{x}(t)$. Then $\mathbf{x}_1 \in K \subset E$ since K is compact. Next define the function $\mathbf{u}(t)$ on $(\alpha, \beta]$ by

$$\mathbf{u}(t) = \begin{cases} \mathbf{x}(t) & \text{for} \quad t \in (\alpha, \beta) \\ \mathrm{x}_1 & \text{for} \quad t = \beta. \end{cases}$$

Then $\mathbf{u}(t)$ is differentiable on $(\alpha, \beta]$. Indeed,

$$\mathrm{u}(t) = \mathrm{x}_0 + \int_0^t \mathrm{f}(\mathrm{u}(s)) ds$$

which implies that

$$\mathbf{u}'(\beta) = \mathbf{f}(\mathbf{u}(\beta));$$

i.e., $\mathbf{u}(t)$ is a solution of the initial value problem (1) on $(\alpha, \beta]$. The function $\mathbf{u}(t)$ is called the continuation of the solution $\mathbf{x}(t)$ to $(\alpha, \beta]$. Since $\mathbf{x}_1 \in E$, it follows from the fundamental existence-uniqueness theorem in Section 2.2 that the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ together with $\mathbf{x}(\beta) = \mathbf{x}_1$ has a unique solution $\mathbf{x}_1(t)$ on some interval $(\beta - a, \beta + a)$. By the above lemma, $\mathbf{x}_1(t) = u(t)$ on $(\beta - a, \beta)$ and $\mathbf{x}_1(\beta) = u(\beta) = \mathbf{x}_1$. So if we define

$$\mathbf{v}(t) = \begin{cases} \mathbf{u}(t) & \text{for} \quad t \in (\alpha, \beta] \\ \mathbf{x}_1(t) & \text{for} \quad t \in [\beta, \beta + a), \end{cases}$$

then $\mathbf{v}(t)$ is a solution of the initial value problem (1) on $(\alpha, \beta + a)$. But this contradicts the fact that (α, β) is the maximal interval of existence for the initial value problem (1). Hence, if $\beta < \infty$, it follows that there exists a $t \in (\alpha, \beta)$ such that $\mathbf{x}(t) \notin K$. If (α, β) is the maximal interval of existence for the initial value problem (1) then $0 \in (\alpha, \beta)$ and the intervals $[0, \beta)$ and $(\alpha, 0]$ are called the right and left maximal intervals of existence respectively. Essentially the same proof yields the following result.

Theorem 3. Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$, let $\mathbf{f} \in C^1(E)$, and let $[0, \beta)$ be the right maximal interval of existence of the solution $\mathbf{x}(t)$ of the initial value problem (1). Assume that $\beta < \infty$. Then given any compact set $K \subset E$, there exists a $t \in (0, \beta)$ such that $\mathbf{x}(t) \notin K$.

Corollary 1. Under the hypothesis of the above theorem, if $\beta < \infty$ and if $\lim_{t \to \beta^-} \mathbf{x}(t)$ exists then $\lim_{t \to \beta^-} \mathbf{x}(t) \in \dot{E}$.

Proof. If $\mathbf{x}_1 = \lim_{t \to \beta^-} \mathbf{x}(t)$, then the function

$$\mathbf{u}(t) = \begin{cases} \mathbf{x}(t) & \text{for} \quad t \in [0, \beta) \\ \mathbf{x}_1 & \text{for} \quad t = \beta \end{cases}$$

is continuous on $[0, \beta]$. Let K be the image of the compact set $[0, \beta]$ under the continuous map $\mathbf{u}(t)$; i.e.,

$$K = \{\mathbf{x} \in \mathbf{R^n} \mid \mathbf{x} = \mathbf{u}(t) \text{ for some } t \in [0, \beta]\}.$$

Then K is compact. Assume that $\mathbf{x}_1 \in E$. Then $K \subset E$ and it follows from Theorem 3 that there exists a $t \in (0, \beta)$ such that $\mathbf{x}(t) \notin K$. This is a contradiction and therefore $\mathbf{x}_1 \notin E$. But since $\mathbf{x}(t) \in E$ for all $t \in [0, \beta)$, it follows that $\mathbf{x}_1 = \lim_{t \to \beta^-} \mathbf{x}(t) \in \bar{E}$. Therefore $\mathbf{x}_1 \in \bar{E} \sim E$; i.e., $\mathbf{x}_1 \in \dot{E}$.

Corollary 2. Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$, let $\mathbf{f} \in C^1(E)$, and let $[0, \beta)$ be the right maximal interval of existence of the solution $\mathbf{x}(t)$ of the initial value problem (1). Assume that there exists a compact set $K \subset E$ such that

$$\{\mathbf{y} \in \mathbf{R^n} \mid \mathbf{y} = \mathbf{x}(t) \text{ for some } t \in [0, \beta)\} \subset K.$$

It then follows that $\beta = \infty$; i.e. the initial value problem (1) has a solution $\mathbf{x}(t)$ on $[0, \infty)$.

Proof. This corollary is just the contrapositive of the statement in Theorem 3.

We next prove the following theorem which strengthens the result on uniform convergence with respect to initial conditions in Remark 3 of Section 2.3.

Theorem 4. Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$ and let $\mathbf{f} \in C^1(E)$. Suppose that the initial value problem (1) has a solution $\mathbf{x}(t, \mathbf{x}_0)$ defined on a closed interval $[a, b]$. Then there exists a $\delta > 0$ and a positive constant K such that for all $\mathbf{y} \in N_\delta(\mathbf{x}_0)$ the initial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\ \dot{\mathbf{x}}(0) &= \mathbf{y} \end{aligned} \tag{2}$$

has a unique solution $\mathbf{x}(t, \mathbf{y})$ defined on $[a, b]$ which satisfies

$$\left|\mathbf{x}(t, \mathbf{y}) - \mathbf{x}(t, \mathbf{x}_0)\right| \leq \left|\mathbf{y} - \mathbf{x}_0\right| e^{K|t|}$$

and

$$\lim_{\mathbf{y} \to \mathbf{x}_0} \mathbf{x}(t, \mathbf{y}) = \mathbf{x}(t, \mathbf{x}_0)$$

uniformly for all $t \in [a, b]$.

Remark 1. If in Theorem 4 we have a function $f(x, \mu)$ depending on a parameter $\mu \in \mathbf{R}^m$ which satisfies f $\in C^1(E)$ where E is an open subset of $\mathbf{R}^{n+m}$ containing ($\mathbf{x}_0, \mu_0$), it can be shown that if for $\boldsymbol{\mu} = \mu_0$ the initial value problem (1) has a solution $\mathbf{x}(t, \mathbf{x}_0, \mu_0)$ defined on a closed interval $a \leq t \leq b$, then there is a $\delta > 0$ and a $K > 0$ such that for all y $\in N_\delta(\mathrm{x}_0)$ and $\mu \in N_\delta(\mu_0)$ the initial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}) \\ \mathbf{x}(0) &= \mathbf{y} \end{aligned}$$

has a unique solution $\mathbf{x}(t, \mathbf{y}, \boldsymbol{\mu})$ defined for $a \leq t \leq b$ which satisfies

$$\left|\mathbf{x}(t, \mathbf{y}, \boldsymbol{\mu}) - \mathbf{x}(t, \mathbf{x}_0, \mu_0)\right| \leq [\![\left|\mathbf{y} - \mathbf{x}_0\right| + \left|\boldsymbol{\mu} - \boldsymbol{\mu}_0\right|]\!] e^{K|t|}$$

and

$$\lim_{(\mathbf{y}, \boldsymbol{\mu}) \to (\mathrm{x}_0, \mu_0)} \mathbf{x}(t, \mathbf{y}, \boldsymbol{\mu}) = \mathbf{x}(t, \mathbf{x}_0, \mu_0)$$

uniformly for all $t \in [a, b]$. In order to prove this theorem, we first establish the following lemma.

Lemma 2. Let E be an open subset of $\mathbf{R}^n$ and let A be a compact subset of E. Then if $\mathrm{f} : E \to \mathbf{R}^n$ is locally Lipschitz on E, it follows that $\mathbf{f}$ satisfies a Lipschitz condition on A.

Proof. Let M be the maximal value of the continuous function $\mathbf{f}$ on the compact set A. Suppose that f does not satisfy a Lipschitz condition on A. Then for every $K > 0$, we can find $\mathbf{x}, \mathbf{y} \in A$ such that

$$|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| > K|\mathbf{y} - \mathbf{x}|.$$

In particular, there exist sequences $\mathbf{x}_n$ and $\mathbf{y}_n$ in A such that

$$\left|\mathbf{f}(\mathbf{y}_n) - \mathbf{f}(\mathbf{x}_n)\right| > n\left|\mathbf{y}_n - \mathbf{x}_n\right| \tag{*}$$

for $n = 1, 2, 3, \ldots$. Since A is compact, there are convergent subsequences, call them $\mathbf{x}_n$ and $\mathbf{y}_n$ for simplicity in notation, such that $\mathbf{x}_n \to \mathbf{x}^*$ and $\mathbf{y}_n \to \mathbf{y}^*$ with $\mathbf{x}^*$ and $\mathbf{y}^*$ in A. It follows that $\mathbf{x}^* = \mathbf{y}^*$ since for all $n = 1, 2, 3, \ldots$

$$\left|\mathbf{y}^* - \mathbf{x}^*\right| = \lim_{n\to\infty}\left|\mathbf{y}_n - \mathbf{x}_n\right| \le \frac{1}{n}\left|\mathbf{f}(\mathbf{y}_n) - \mathbf{f}(\mathbf{x}_n)\right| \le \frac{2M}{n}.$$

Now, by hypotheses, there exists a neighborhood N_0 of $\mathbf{x}^*$ and a constant K_0 such that f satisfies a Lipschitz condition with Lipschitz constant K_0 for all $\mathbf{x}$ and $\mathbf{y} \in N_0$. But since $\mathbf{x}_n$ and $\mathbf{y}_n$ approach $\mathbf{x}^*$ as $n \to \infty$, it follows that $\mathbf{x}_n$ and $\mathbf{y}_n$ are in N_0 for n sufficiently large; i.e., for n sufficiently large

$$\left|\mathbf{f}(\mathbf{y}_n) - \mathbf{f}(\mathbf{x}_n)\right| \le K\left|\mathbf{y}_n - \mathbf{x}_n\right|.$$

But for $n \ge K$, this contradicts the above inequality (*) and this establishes the lemma.

Proof (of Theorem 4). Since $[a, b]$ is compact and $\mathbf{x}(t, x_0)$ is a continuous function of t, $\{\mathbf{x} \in \mathbf{R^n} \mid \mathbf{x} = \mathbf{x}(t, \mathbf{x}_0)$ and $a \le t \le b\}$ is a compact subset of E. And since E is open, there exists an $\varepsilon > 0$ such that the compact set

$$A = \{\mathbf{x} \in \mathbf{R}^n | |\mathbf{x} - \mathbf{x}(t, \mathbf{x}_0)| \le \varepsilon \text{ and } a \le t \le b\}$$

is a subset of E. Since $\mathrm{f} \in C^1(E)$, it follows from the lemma in Section 2.2 that f is locally Lipschitz in E; and then by the above lemma, f satisfies a Lipschitz condition

$$|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| \le K|\mathbf{y} - \mathbf{x}|$$

for all $\mathbf{x}, \mathbf{y} \in A$. Choose $\delta > 0$ so small that $\delta \le \varepsilon$ and $\delta \le \varepsilon e^{-K(b-a)}$. Let $\mathbf{y} \in N_\delta(\mathbf{x}_0)$ and let $\mathbf{x}(t, \mathbf{y})$ be the solution of the initial value problem (2) on its maximal interval of existence (α, β). We shall show that $[a, b] \subset (\alpha, \beta)$. Suppose that $\beta \le b$. It then follows that $\mathbf{x}(t, \mathbf{y}) \in A$ for all $t \in (\alpha, \beta)$ because if this were not true then there would exist a $t^* \in (\alpha, \beta)$ such that $\mathbf{x}(t, \mathbf{x}_0) \in A$ for $t \in (\alpha, t^*]$ and $\mathbf{x}(t^*, \mathbf{y}) \in \dot{A}$. But then

$$\begin{aligned}\left|\mathbf{x}(t,\mathbf{y})-\mathbf{x}(t,\mathbf{x}_0)\right| &\leq \left|\mathbf{y}-\mathbf{x}_0\right| + \int_0^t \mid \mathbf{f}(\mathbf{x}(s,\mathbf{y})) - f(\mathbf{x}(s,\mathbf{x}_0) \mid ds \\ &\leq \left|\mathbf{y}-\mathbf{x}_0\right| + K\int_0^t \left|\mathbf{x}(s,\mathbf{y})-\mathbf{x}(s,\mathbf{x}_0)\right| ds\end{aligned}$$

for all $t \in (\alpha, t^*]$. And then by Gronwall's Lemma in Section 2.3, it follows that

$$\left|\mathbf{x}(t^*,\mathbf{y})-\mathbf{x}(t^*,\mathbf{x}_0)\right| \leq \left|\mathbf{y}-\mathbf{x}_0\right| e^{K|t^*|} < \delta e^{K(b-a)} < \varepsilon$$

since $t^* < \beta \leq b$. Thus $\mathbf{x}(t^*, \mathbf{y})$ is an interior point of A, a contradiction. Thus, $\mathbf{x}(t, y) \in A$ for all $t \in (\alpha, \beta)$. But then by Theorem 2, (α, β) is not the maximal interval of existence of $\mathbf{x}(t, \mathbf{y})$, a contradiction. Thus $b < \beta$. It is similarly proved that $\alpha < a$. Hence, for all $y \in N_\delta(x_0)$, the initial value problem (2) has a unique solution defined on $[a, b]$. Furthermore, if we assume that there is a $t^* \in [a, b)$ such that $\mathbf{x}(t, \mathbf{y}) \in A$ for all $t \in [a, t^*)$ and $\mathbf{x}(t^*, \mathbf{y}) \in \dot{A}$, a repeat of the above argument based on Gronwall's Lemma leads to a contradiction and shows that $\mathbf{x}(t, \mathbf{y}) \in A$ for all $t \in [a, b]$ and hence that

$$\left|\mathbf{x}(t,\mathbf{y})-\mathbf{x}(t,\mathbf{x}_0)\right| \leq \left|\mathbf{y}-\mathbf{x}_0\right| e^{K|t|}$$

for all $t \in [a, b]$. It then follows that

$$\lim_{\mathbf{y}\to\mathbf{x}_0} \mathbf{x}(t,\mathbf{y}) = \mathbf{x}(t,\mathbf{x}_0)$$

uniformly for all $t \in [a, b]$.

1.4 The Flow Defined by a Differential Equation

In Section 1.9 of Chapter 1, we defined the flow, $\mathrm{e}^{At} : \mathbf{R}^n \to \mathbf{R}^n$, of the linear system

$$\dot{\mathbf{x}} = A\mathbf{x}.$$

The mapping $\phi_t = \mathrm{e}^{\mathrm{A}t}$ satisfies the following basic properties for all $\mathrm{x} \in \mathbf{R}^n$:

(i) $\phi_0(\mathbf{x}) = \mathbf{x}$

(ii) $\phi_s\left(\phi_t(\mathbf{x})\right) = \phi_{s+t}(\mathbf{x})$ for all $s, t \in \mathbf{R}$

(iii) $\phi_{-t}\left(\phi_t(\mathbf{x})\right) = \phi_t\left(\phi_{-t}(\mathbf{x})\right) = \mathbf{x}$ for all $t \in \mathbf{R}$.

Property (i) follows from the definition of $e^{\mathrm{A}t}$, property (ii) follows from Proposition 2 in Section 1.3 of Chapter 1, and property (iii) follows from Corollary 2 in Section 1.3 of Chapter 1. In this section, we define the flow, ϕ_t, of the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

and show that it satisfies these same basic properties. In the following definition, we denote the maximal interval of existence (α, β) of the solution $\phi(t, \mathbf{x}_0)$ of the initial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned} \tag{2}$$

by $I(\mathrm{x}_0)$ since the endpoints α and β of the maximal interval generally depend on x_0; cf. problems 1(a) and (d) in Section 2.4.

Definition 1. Let E be an open subset of $\mathbf{R}^n$ and let $\mathbf{f} \in C^1(E)$. For $\mathbf{x}_0 \in E$, let $\phi(t, \mathbf{x}_0)$ be the solution of the initial value problem (2) defined on its maximal interval of existence $I(\mathrm{x}_0)$. Then for $t \in I(\mathrm{x}_0)$, the set of mappings ϕ_t defined by

$$\phi_t(\mathbf{x}_0) = \phi(t, \mathbf{x}_0)$$

is called the flow of the differential equation (1) or the flow defined by the differential equation (1); ϕ_t is also referred to as the flow of the vector field $\mathbf{f}(\mathbf{x})$.

If we think of the initial point x_0 as being fixed and let $I = I(x_0)$, then the mapping $\phi(\cdot, \mathbf{x}_0) : I \to E$ defines a solution curve or trajectory of the system (1) through the point $x_0 \in E$. As usual, the mapping $\phi(\cdot, x_0)$ is identified with its graph in $I \times E$ and a trajectory is visualized as a motion along a curve Γ through the point x_0 in the subset E of the phase space $\mathbf{R}^n$; cf. Figure 1. On the other hand, if we think of the point x_0 as varying throughout $K \subset E$, then the flow of the differential equation (1), $\phi_t : K \to E$ can be viewed as the motion of all the points in the set K; cf. Figure 2.

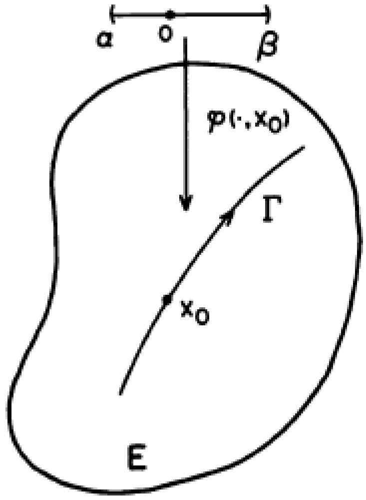

Figure 1. A trajectory Γ of the system (1).

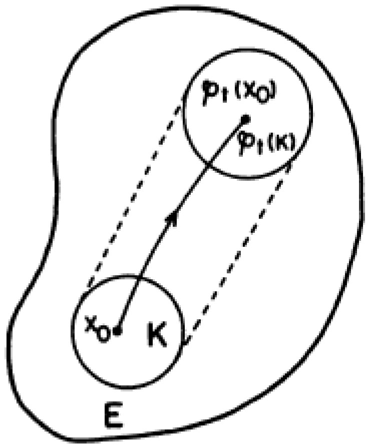

Figure 2. The flow ϕ_t of the system (1).

If we think of the differential equation (1) as describing the motion of a fluid, then a trajectory of (1) describes the motion of an individual particle in the fluid while the flow of the differential equation (1) describes the motion of the entire fluid. We now show that the basic properties (i)-(iii) of linear flows are also satisfied by nonlinear flows. But first we extend Theorem 1 of Section 2.3, establishing that $\phi\,(t, x_0)$ is a locally smooth function, to a global result. Using the same notation as in Definition 1, let us define the set $\Omega \subset \mathbf{R} \times E$ as

$$\Omega = \{(t, x_0) \in \mathbf{R} \times E \mid t \in I\,(\mathbf{x}_0)\}\,.$$

Theorem 1. Let E be an open subset of $\mathbf{R}^n$ and let $\mathbf{f} \in C^1(E)$. Then Ω is an open subset of $\mathbf{R} \times E$ and $\phi \in C^1(\Omega)$.

Proof. If $(t_0, x_0) \in \Omega$ and $t_0 > 0$, then according to the definition of the set Ω, the solution $\mathbf{x}(t) = \phi\,(t, x_0)$ of the initial value problem (2) is defined on $[0, t_0]$. Thus, as in the proof of Theorem 2 in Section 2.4, the solution $\mathbf{x}(t)$ can be extended to an interval $[0, t_0 + \varepsilon]$ for some $\varepsilon > 0$; i.e., $\phi\,(t, x_0)$ is defined on the closed interval $[t_0 - \varepsilon, t_0 + \varepsilon]$. It then follows from Theorem 4 in Section 2.4 that there exists a neighborhood of $\mathrm{x}_0, N_\delta\,(\mathrm{x}_0)$, such that

$\phi(t, \mathbf{y})$ is defined on $[t_0 - \varepsilon, t_0 + \varepsilon] \times N_\delta\,(\mathbf{x}_0)$; i.e., $(t_0 - \varepsilon, t_0 + \varepsilon) \times N_\delta\,(\mathbf{x}_0) \subset \Omega$. Therefore, Ω is open in $R \times E$. It follows from Theorem 4 in Section 2.4 that $\phi \in C^1(G)$ where $G = (t_0 - \varepsilon, t_0 + \varepsilon) \times N_\delta\,(\mathbf{x}_0)$. A similar proof holds for $t_0 \leq 0$, and since $(\,t_0, \mathrm{x}_0\,)$ is an arbitrary point in Ω, it follows that $\phi \in C^1(\Omega)$.

Remark. Theorem 1 can be generalized to show that if $\mathrm{f} \in C^r(E)$ with $r \geq 1$, then $\phi \in C^r(\Omega)$ and that if f is analytic in E, then ϕ is analytic in Ω.

Theorem 2. Let E be an open set of $\mathbf{R}^n$ and let $\mathbf{f} \in C^1(E)$. Then for all $\mathrm{x}_0 \in E$, if $t \in I(\mathrm{x}_0)$ and $s \in I\left(\phi_t(\mathrm{x}_0)\right)$, it follows that $s + t \in I(\mathrm{x}_0)$ and

$$\phi_{s+t}(\mathrm{x}_0) = \phi_s\left(\phi_t(\mathrm{x}_0)\right).$$

Proof. Suppose that $s > 0, t \in I(\mathrm{x}_0)$ and $s \in I\left(\phi_t(\mathrm{x}_0)\right)$. Let the maximal interval $I(\mathbf{x}_0) = (\alpha, \beta)$ and define the function $\mathbf{x} : (\alpha, s + t] \to E$ by

$$\mathbf{x}(r) = \begin{cases} \phi(r, \mathrm{x}_0) & \text{if } \alpha < r \le t \\ \phi\left(r - t, \phi_t(\mathrm{x}_0)\right) & \text{if } t \le r \le s + t. \end{cases}$$

Then $\mathbf{x}(r)$ is a solution of the initial value problem (2) on $(\alpha, s+t)$. Hence $s+t \in I(\mathbf{x}_0)$ and by uniqueness of solutions

$$\phi_{s+t}(\mathrm{x}_0) = \mathbf{x}(s + t) = \phi\left(s, \phi_t(\mathrm{x}_0)\right) = \phi_s\left(\phi_t(\mathrm{x}_0)\right).$$

If $s = 0$ the statement of the theorem follows immediately. And if $s < 0$, then we define the function $\mathbf{x} : [s + t, \beta) \to E$ by

$$\mathbf{x}(t) = \begin{cases} \phi(r, \mathbf{x}_0) & \text{if } t \le r < \beta \\ \phi\left(r - t, \phi_t(\mathrm{x}_0)\right) & \text{if } s + t \le r \le t. \end{cases}$$

Then $\mathbf{x}(r)$ is a solution of the initial value problem (2) on $[s + t, \beta)$ and the last statement of the theorem follows from the uniqueness of solutions as above.

Theorem 3. Under the hypotheses of Theorem 1, if $(t, x_0) \in \Omega$ then there exists a neighborhood U of x_0 such that $\{t\} \times U \subset \Omega$. It then follows that the set $V = \phi_t(U)$ is open in E and that

$$\phi_{-t}\left(\phi_t(\mathbf{x})\right) = \mathbf{x} \text{ for all } \mathbf{x} \in U$$

and

$$\phi_t\left(\phi_{-t}(\mathbf{y})\right) = \mathbf{y} \text{ for all } \mathbf{y} \in V.$$

Proof. If $(t, x_0) \in \Omega$ then if follows as in the proof of Theorem 1 that there exists a neighborhood of $\mathrm{x}_0, U = N_\delta(\mathrm{x}_0)$, such that $(t - \varepsilon, t + \varepsilon) \times U \subset \Omega$; thus, $\{t\} \times U \subset \Omega$. For $\mathbf{x} \in U$, let $\mathbf{y} = \phi_t(\mathbf{x})$ for all $t \in I(\mathbf{x})$. Then $-t \in I(\mathbf{y})$ since the function $\mathrm{h}(s) = \phi(s + t, \mathbf{y})$ is a solution of (1) on $[-t, 0]$ that satisfies $\mathbf{h}(-t) = \mathbf{y}$; i.e., ϕ_{-t} is defined on the set $V = \phi_t(U)$. It then follows from Theorem 2 that $\phi_{-t}\left(\phi_t(\mathbf{x})\right) = \phi_0(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in U$ and that $\phi_t\left(\phi_{-t}(\mathbf{y})\right) = \phi_0(\mathbf{y}) = \mathbf{y}$ for all $\mathbf{y} \in V$. It remains to prove that V is open. Let $V^* \supset V$ be the maximal subset of E on which ϕ_{-t} is defined. V^* is open because Ω is open and $\phi_{-t} : V^* \to E$ is continuous because by Theorem 1, ϕ is continuous. Therefore, the inverse image of the open set U under the continuous map ϕ_{-t}, i.e., $\phi_t(U)$, is open in E. Thus, V is open in E.

In Chapter 3 we show that the time along each trajectory of (1) can be rescaled, without affecting the phase portrait of (1), so that for all $\mathbf{x}_0 \in E$, the solution $\phi(t, x_0)$ of the initial value problem (2) is defined for all $t \in \mathbf{R}$; i.e., for all $x_0 \in E, I(x_0) = (-\infty, \infty)$. This rescaling avoids some of the complications found in stating the above theorems. Once this rescaling has been made, it follows that $\Omega = \mathbf{R} \times E, \phi \in C^1(\mathbf{R} \times E), \phi_t \in C^1(E)$ for all $t \in \mathbf{R}$, and properties (i)-(iii) for the flow of the nonlinear system (1) hold for all $t \in \mathbf{R}$ and $\mathbf{x} \in E$ just as for the linear flow e^{At}. In the remainder of this chapter, and in particular in Sections 2.7 and 2.8 of this chapter, it will be assumed that this rescaling has been made so that for all $x_0 \in E, \phi(t, x_0)$ is defined for all $t \in \mathbf{R}$; i.e., we shall assume throughout the remainder of this chapter that the flow of the nonlinear system (1) $\phi_t \in C^1(E)$ for all $t \in \mathbf{R}$.

Definition 2. Let E be an open subset of $\mathbf{R}^n$, let $f \in C^1(E)$, and let $\phi_t : E \to E$ be the flow of the differential equation (1) defined for all $t \in \mathbf{R}$. Then a set $S \subset E$ is called invariant with respect to the flow ϕ_t if $\phi_t(S) \subset S$ for all $t \in \mathbf{R}$ and S is called positively (or negatively) invariant with respect to the flow ϕ_t if $\phi_t(S) \subset S$ for all $t \geq 0$ (or $t \leq 0$). In Section 1.9 of Chapter 1 we showed that the stable, unstable and center subspaces of the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ are invariant under the linear flow $\phi_t = e^{At}$. A similar result is established in Section 2.7 for the nonlinear flow ϕ_t of (1).

1.5 Linearization

A good place to start analyzing the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

is to determine the equilibrium points of (1) and to describe the behavior of (1) near its equilibrium points. In the next two sections it is shown that the local behavior of the nonlinear system (1) near a hyperbolic equilibrium point x_0 is qualitatively determined by the behavior of the linear system

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{2}$$

with the matrix $A = D\mathbf{f}(\mathbf{x}_0)$, near the origin. The linear function $A\mathbf{x} = \mathrm{D}\,\mathbf{f}(\mathbf{x}_0)\,\mathbf{x}$ is called the linear part of $\mathbf{f}$ at $\mathbf{x}_0$.

Definition 1. A point $x_0 \in \mathbf{R}^n$ is called an equilibrium point or critical point of (1) if $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$. An equilibrium point $\mathbf{x}_0$ is called a hyperbolic equilibrium point of (1) if none of the eigenvalues of the matrix $D\mathbf{f}(x_0)$ have zero real part. The linear system (2) with the matrix $A = Df(x_0)$ is called the linearization of

(1) at $\mathbf{x}_0$.

If $x_0 = 0$ is an equilibrium point of (1), then $\mathbf{f}(0) = 0$ and, by Taylor's Theorem,

$$\mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{0})\mathbf{x} + \frac{1}{2}D^2\mathbf{f}(\mathbf{0})(\mathbf{x}, \mathbf{x}) + \cdots .$$

It follows that the linear function $D\mathbf{f}(\mathbf{0})\mathbf{x}$ is a good first approximation to the nonlinear function $\mathbf{f}(\mathbf{x})$ near $\mathbf{x} = \mathbf{0}$ and it is reasonable to expect that the behavior of the nonlinear system (1) near the point $\mathbf{x} = 0$ will be approximated by the behavior of its linearization at $\mathbf{x} = \mathbf{0}$. In Section 2.7 it is shown that this is indeed the case if the matrix $Df(0)$ has no zero or pure imaginary eigenvalues.

Note that if x_0 is an equilibrium point of (1) and $\phi_t : E \to \mathbf{R}^n$ is the flow of the differential equation (1), then $\phi_t(\mathbf{x}_0) = \mathbf{x}_0$ for all $t \in \mathbf{R}$. Thus, $\mathbf{x}_0$ is called a fixed point of the flow ϕ_t; it is also called a zero, a critical point, or a singular point of the vector field $\mathbf{f} : E \to \mathbf{R}^n$. We next give a rough classification of the equilibrium points of (1) according to the signs of the real parts of the eigenvalues of the matrix $Df(\mathrm{x}_0)$. A finer classification is given in Section 2.10 for planar vector fields.

Definition 2. An equilibrium point x_0 of (1) is called a sink if all of the eigenvalues of the matrix $D\mathbf{f}(\mathrm{x}_0)$ have negative real part; it is called a source if all of the eigenvalues of $D\mathbf{f}(\mathrm{x}_0)$ have positive real part; and it is called a saddle if it is a hyperbolic equilibrium point and $Df(\mathrm{x}_0)$ has at least one eigenvalue with a positive real part and at least one with a negative real part.

1.6 The Stable Manifold Theorem

The stable manifold theorem is one of the most important results in the local qualitative theory of ordinary differential equations. The theorem shows that near a hyperbolic equilibrium point x_0, the nonlinear system (1) has stable and unstable manifolds S and U tangent at x_0 to the stable and unstable subspaces E^s and E^u of the linearized system (2) where $A = D\mathbf{f}(\mathbf{x}_0)$. Furthermore, S and U are of the same dimensions as E^a and E^u, and if ϕ_t is the flow of the nonlinear system (1), then S and U are positively and negatively invariant under ϕ_t respectively and satisfy

$$\lim_{t\to\infty} \phi_t(\mathbf{c}) = \mathrm{x}_0 \text{ for all } \mathbf{c} \in S$$

and

$$\lim_{t\to-\infty} \phi_t(\mathbf{c}) = \mathbf{x}_0 \text{ for all } \mathbf{c} \in U.$$

We first illustrate these ideas with an example and then make them more precise by proving the stable manifold theorem. It is assumed that the equilibrium point $\mathbf{x}_0$ is located at the origin throughout the remainder of this section. If this is not the case, then the equilibrium point x_0 can be translated to the origin by the affine transformation of coordinates $x \to \mathbf{x} - \mathbf{x_0}$.

Example 1. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= -x_2 + x_1^2\\ \dot{x}_3 &= x_3 + x_1^2.\end{aligned}$$

The only equilibrium point of this system is at the origin. The matrix

$$A = D\mathbf{f}(0) = \begin{bmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Thus, the stable and unstable subspaces E^s and E^u of (2) are the x_1, x_2 plane and the x_3-axis respectively. After solving the first differential equation, $\dot{x}_1 = -x_1$, the nonlinear system reduces to two uncoupled first-order linear differential equations which are easily solved. The solution is given by

$$\begin{aligned}x_1(t) &= c_1 e^{-t}\\ x_2(t) &= c_2 e^{-t} + c_1^2\left(e^{-t} - e^{-2t}\right)\\ x_3(t) &= c_3 e^{t} + \frac{c_1^2}{3}\left(e^{t} - e^{-2t}\right)\end{aligned}$$

where $\mathbf{c} = \mathbf{x}(0)$. Clearly, $\lim_{t\to\infty} \phi_t(\mathbf{c}) = \mathbf{0}$ iff $c_3 + c_1^2/3 = 0$. Thus,

$$S = \left\{\mathbf{c} \in \mathbf{R}^3 \mid c_3 = -c_1^2/3\right\}.$$

Similarly, $\lim_{t\to-\infty} \phi_t(c) = 0$ iff $c_1 = c_2 = 0$ and therefore

$$U = \left\{c \in \mathbf{R}^3 \mid c_1 = c_2 = 0\right\}$$

The stable and unstable manifolds for this system are shown in Figure 1. Note that the surface S is tangent to E^s, i.e., to the x_1, x_2 plane at the origin and that $U = E^u$.

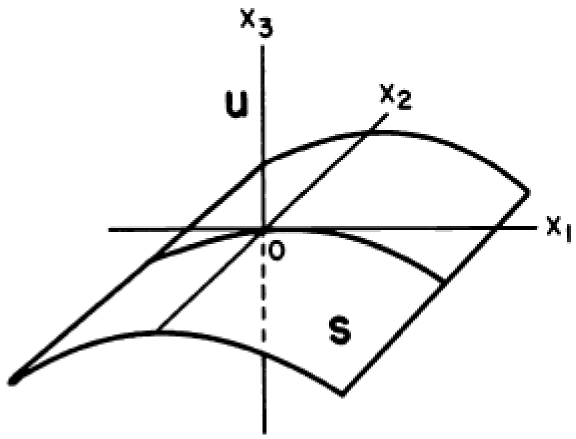

Figure 1

Before proving the stable manifold theorem, we first define the concept of a smooth surface or differentiable manifold.

Definition 1. Let X be a metric space and let A and B be subsets of X. A homeomorphism of A onto B is a continuous one-to-one map of A onto $B, h : A \to B$, such that $h^{-1} : B \to A$ is continuous. The sets A and B are called homeomorphic or topologically equivalent if there is a homeomorphism of A onto B. If we wish to emphasize that h maps A onto B.

Definition 2. An n-dimensional differentiable manifold, M (or a manifold of class C^k), is a connected metric space with an open covering $\{U_\alpha\}$, i.e., $M = U_\alpha U_\alpha$, such that

(1) for all α, U_α is homeomorphic to the open unit ball in $\mathbf{R}^n, B = \{\mathbf{x} \in \mathbf{R}^n \| \mathbf{x} |< 1\}$, i.e., for all α there exists a homeomorphism of U_α onto $B, \mathbf{h}_\alpha : U_\alpha \to B$, and

(2) if $U_\alpha \cap U_\beta \neq \emptyset$ and $\mathbf{h}_\alpha : U_\alpha \to B, \mathbf{h}_\beta : U_\beta \to B$ are homeomorphisms, then $\mathbf{h}_\alpha \left(U_\alpha \cap U_\beta\right)$ and $\mathbf{h}_\beta \left(U_\alpha \cap U_\beta\right)$ are subsets of $\mathbf{R}^n$ and the map

$$\mathbf{h} = \mathbf{h}_\alpha \circ \mathbf{h}_\beta^{-1} : \mathbf{h}_\beta \left(U_\alpha \cap U_\beta\right) \to \mathbf{h}_\alpha \left(U_\alpha \cap U_\beta\right)$$

is differentiable (or of class C^k) and for all $\mathbf{x} \in \mathbf{h}_\beta \left(U_\alpha \cap U_\beta\right)$, the Jacobian determinant $\det D\mathbf{h}(\mathbf{x}) \neq 0$. The manifold M is said to be analytic if the maps $h = h_\alpha \circ h_\beta^{-1}$ are analytic.

The cylindrical surface S in the above example is a two-dimensional differentiable manifold. The projection of the x_1, x_2 plane onto S maps the unit disks centered at the points (m, n) in the $x_1 x_2$ plane onto homeomorphic images of the unit disk $B = \left\{\mathbf{x} \in \mathbf{R}^2 \mid x_1^2 + x_2^2 < 1\right\}$. These sets $U_{mn} \subset S$ then form a countable open cover of S in this case.

The pair ($U_\alpha, \mathbf{h}_\alpha$) is called a chart for the manifold M and the set of all charts is called an atlas for M. The differentiable manifold M is called orientable if there is an atlas with $\det D\mathbf{h}_\alpha \circ \mathbf{h}_\beta^{-1}(\mathbf{x}) > 0$ for all α, β and $\mathbf{x} \in \mathbf{h}_\beta \left(U_\alpha \cap U_\beta\right)$.

Theorem (The Stable Manifold Theorem). Let E be an open subset of $\mathbf{R}^n$ containing the origin, let $\mathbf{f} \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system (1). Suppose that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and that $D\mathbf{f}(\mathbf{0})$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linear system (2) at 0 such that for all $t \geq 0, \phi_t(S) \subset S$ and for all $\mathrm{x}_0 \in S$.

$$\lim_{t\to\infty} \phi_t(\mathbf{x}_0) = \mathbf{0}$$

and there exists an $n-k$ dimensional differentiable manifold U tangent to the unstable subspace E^u of (2) at 0 such that for all $t \leq 0, \phi_t(U) \subset U$ and for all $\mathbf{x}_0 \in U$,

$$\lim_{t\to-\infty} \phi_t(\mathbf{x}_0) = \mathbf{0}.$$

Before proving this theorem, we remark that if $\mathrm{f} \in C^1(E)$ and $f(0) = 0$, then the system (1) can be written as

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{F}(\mathbf{x}) \tag{3}$$

where $A = D\mathbf{f}(\mathbf{0}), \mathbf{F}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - A\mathbf{x}, \mathbf{F} \in C^1(E), \mathbf{F}(\mathbf{0}) = \mathbf{0}$ and $D\mathbf{F}(0) = 0$. This in turn implies that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|x| \leq \delta$ and $|\mathbf{y}| \leq \delta$ imply that

$$|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})| \leq \varepsilon|\mathbf{x} - \mathbf{y}|. \tag{4}$$

Furthermore, as in Section 1.8 of Chapter 1, there is an $n \times n$ invertible matrix C such that

$$B = C^{-1}AC = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

where the eigenvalues $\lambda_1, \ldots, \lambda_k$ of the $k \times k$ matrix P have negative real part and the eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ of the $(n-k) \times (n-k)$ matrix Q have postive real part. We can choose $\alpha > 0$ sufficiently small that for $j = 1, \ldots, k$,

$$\mathrm{Re}\left(\lambda_j\right) < -\alpha < 0. \tag{5}$$

Letting $y = C^{-1}x$, the system (3) then has the form

$$\dot{\mathbf{y}} = B\mathbf{y} + \mathbf{G}(\mathbf{y}) \tag{6}$$

where $\mathbf{G}(\mathbf{y}) = C^{-1}\mathbf{F}(C\mathbf{y}) \in C^1(\tilde{E})$ where $\tilde{E} = C^{-1}(E)$ and $\mathbf{G}$ satisfies the Lipschitz-type condition (4) above.

It will be shown in the proof that there are $n-k$ differentiable functions $\psi_j(y_1,\ldots,y_k)$ such that the equations

$$y_j = \psi_j(y_1,\ldots,y_k),\, j = k+1,\ldots,n$$

define a k-dimensional differentiable manifold $\tilde{S}$ in $\mathbf{y}$-space. The differentiable manifold S in $\mathbf{x}$-space is then obtained from $\tilde{S}$ under the linear transformation of coordinates $\mathbf{x} = C\mathbf{y}$.

Proof. Consider the system (6). Let

$$U(t) = \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } V(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{Qt} \end{bmatrix}.$$

Then $\dot{U} = BU$, $\dot{V} = BV$ and

$$e^{Bt} = U(t) + V(t)$$

It is not difficult to see that with $\alpha > 0$ chosen as in (5), we can choose $K > 0$ sufficiently large and $\sigma > 0$ sufficiently small that

$$\|U(t)\| \le Ke^{-(\alpha+\sigma)t} \text{ for all } t \ge 0$$

and

$$\|V(t)\| \le Ke^{\sigma t} \text{ for all } t \le 0.$$

Next consider the integral equation

$$\mathbf{u}(t,\mathbf{a}) = U(t)\mathbf{a} + \int_0^t U(t-s)\mathbf{G}(\mathbf{u}(s,\mathbf{a}))ds - \int_t^\infty V(t-s)\mathbf{G}(\mathbf{u}(s,\mathbf{a}))ds. \tag{7}$$

If $\mathbf{u}(t,\mathbf{a})$ is a continuous solution of this integral equation, then it is a solution of the differential equation (6). We now solve this integral equation by the method of successive approximations. Let

$$\mathbf{u}^{(0)}(t,\mathbf{a}) = \mathbf{0}$$

and

$$\begin{aligned}\mathbf{u}^{(j+1)}(t,\mathbf{a}) =& U(t)\mathbf{a} + \int_0^t U(t-s)\mathbf{G}\left(\mathbf{u}^{(j)}(s,\mathbf{a})\right)ds \\ &- \int_t^\infty V(t-s)\mathbf{G}\left(\mathbf{u}^{(j)}(s,\mathbf{a})\right)ds.\end{aligned} \tag{8}$$

Assume that the induction hypothesis

$$\left|\mathbf{u}^{(j)}(t,\mathbf{a})-\mathbf{u}^{(j-1)}(t,\mathbf{a})\right| \le \frac{K|\mathbf{a}|e^{-\alpha t}}{2^{j-1}} \tag{9}$$

holds for $j = 1, 2, \ldots, m$ and $t \ge 0$. It clearly holds for $j = 1$ provided $t \ge 0$. Then using the Lipschitz-type condition (4) satisfied by the function $\mathbf{G}$ and the above estimates on $\|U(t)\|$ and $\|V(t)\|$, it follows from the induction hypothesis that for $t \ge 0$

$$\begin{aligned}
\left|\mathbf{u}^{(m+1)}(t,\mathbf{a})-\mathbf{u}^{(m)}(t,\mathbf{a})\right| \le & \int_0^t \|U(t-s)\|\varepsilon \left|\mathbf{u}^{(m)}(s,\mathbf{a})-\mathbf{u}^{(m-1)}(s,\mathbf{a})\right| ds \\
& + \int_t^\infty \|V(t-s)\|\varepsilon \left|\mathbf{u}^{(m)}(s,\mathbf{a})-\mathbf{u}^{(m-1)}(s,\mathbf{a})\right| ds \\
\le & \varepsilon \int_0^t K e^{-(\alpha+\sigma)(t-s)} \frac{K|\mathbf{a}|e^{-\alpha s}}{2^{m-1}} ds \\
& + \varepsilon \int_0^\infty K e^{\sigma(t-s)} \frac{K|\mathbf{a}|e^{-\alpha s}}{2^{m-1}} ds \\
\le & \frac{\varepsilon K^2|\mathbf{a}|e^{-\alpha t}}{\sigma 2^{m-1}} + \frac{\varepsilon K^2|\mathbf{a}|e^{-\alpha t}}{\sigma 2^{m-1}} \\
< & \left(\frac{1}{4}+\frac{1}{4}\right)\frac{K|\mathbf{a}|e^{-\alpha t}}{2^{m-1}} = \frac{K|\mathbf{a}|e^{-\alpha t}}{2^m}
\end{aligned} \tag{10}$$

provided $\varepsilon K/\sigma < 1/4$; i.e., provided we choose $\varepsilon < \frac{\sigma}{4K}$. In order that the condition (4) hold for the function $\mathbf{G}$, it suffices to choose $K|\mathbf{a}| < \delta/2$; i.e., we choose $|a| < \frac{\delta}{2K}$. It then follows by induction that (9) holds for all $j = 1, 2, 3, \ldots$ and $t \ge 0$. Thus, for $n > m > N$ and $t \ge 0$,

$$\begin{aligned}
\left|\mathbf{u}^{(\mathbf{n})}(t,\mathbf{a})-\mathbf{u}^{(m)}(t,\mathbf{a})\right| &\le \sum_{j=N}^{\infty} \left|\mathbf{u}^{(j+1)}(t,\mathbf{a})-\mathbf{u}^{(j)}(t,\mathbf{a})\right| \\
&\le K|\mathbf{a}|\sum_{j=N}^{\infty}\frac{1}{2^j} = \frac{K|\mathbf{a}|}{2^{N-1}}.
\end{aligned}$$

This last quantity approaches zero as $N \to \infty$ and therefore $\left\{\mathbf{u}^{(j)}(t,\mathbf{a})\right\}$ is a Cauchy sequence of continuous functions. According to the theorem in Section 2.2 ,

$$\lim_{j\to\infty}\mathbf{u}^{(j)}(t,\mathbf{a}) = \mathbf{u}(t,\mathbf{a})$$

uniformly for all $t \ge 0$ and $|\mathbf{a}| < \delta/2K$. Taking the limit of both sides of (8), it follows from the uniform convergence that the continuous function $\mathbf{u}(t, \mathrm{a})$ satisfies the integral equation (7) and hence the differential equation (6). It follows by induction and the fact that $\mathbf{G} \in C^1(\tilde{E})$ that $\mathbf{u}^{(j)}(t,a)$ is a differentiable function of a for $t \ge 0$ and $|\mathrm{a}| < \delta/2K$. Thus, it follows from the uniform convergence that $\mathbf{u}(t,\mathbf{a})$ is a differentiable function of a for $t \geq 0$ and $|a| < \delta/2K$. The estimate (10) implies that

$$|\mathbf{u}(t, \mathbf{a})| \leq 2K|\mathbf{a}|e^{-\alpha t} \tag{11}$$

for $t \geq 0$ and $|a| < \delta/2K$.

It is clear from the integral equation (7) that the last $n-k$ components of the vector a do not enter the computation and hence they may be taken as zero. Thus, the components $u_j(t, a)$ of the solution $u(t, a)$ satisfy the initial conditions

$$u_j(0, \mathbf{a}) = a_j \text{ for } j = 1, \ldots, k$$

and

$$u_j(0, \mathrm{a}) = -\left(\int_0^\infty V(-s)\mathbf{G}\left(\mathbf{u}\left(s, a_1, \ldots, a_k, 0\right)\right) ds\right)_j \text{ for } j = k+1, \ldots, n.$$

For $j = k+1, \ldots, n$ we define the functions

$$\psi_j\left(a_1, \ldots, a_k\right) = u_j\left(0, a_1, \ldots, a_k, 0, \ldots, 0\right). \tag{12}$$

Then the initial values $y_j = u_j\left(0, a_1, \ldots, a_k, 0, \ldots, 0\right)$ satisfy

$$y_j = \psi_j\left(y_1, \ldots, y_k\right) \text{ for } j = k+1, \ldots, n$$

according to the definition (12). These equations then define a differentiable manifold $\tilde{S}$ for $\sqrt{y_1^2 + \cdots + y_k^2} < \delta/2K$. Furthermore, if $\mathbf{y}(t)$ is a solution of the differential equation (6) with $\mathbf{y}(0) \in \tilde{S}$, i.e., with $\mathbf{y}(0) = \mathbf{u}(0, \mathbf{a})$, then

$$\mathbf{y}(t) = \mathbf{u}(t, \mathbf{a}).$$

It follows from the estimate (11) that if $y(t)$ is a solution of (6) with $\mathbf{y}(0) \in \tilde{S}$, then $\mathbf{y}(t) \to 0$ as $t \to \infty$. It can also be shown that if $\mathbf{y}(t)$ is a solution of (6) with $\mathbf{y}(0) \notin \tilde{S}$ then $\mathbf{y}(t) \nrightarrow 0$ as $t \to \infty$; cf. Coddington and Levinson. It therefore follows from Theorem 2 in Section 2.5 that if $\mathbf{y}(0) \in \tilde{S}$, then $\mathbf{y}(t) \in \tilde{S}$ for all $t \geq 0$. And it can be shown as in [C/L], p. 333 that

$$\frac{\partial \psi_j}{\partial y_i}(0) = 0$$

for $i = 1, \ldots, k$ and $j = k+1, \ldots, n$; i.e., the differentiable manifold $\tilde{S}$ is tangent to the stable subspace $E^s = \{\mathbf{y} \in \mathbf{R}^n \mid y_1 = \cdots y_k = 0\}$ of the linear system $\dot{\mathbf{y}} = B\mathbf{y}$ at 0 .

The existence of the unstable manifold $\tilde{U}$ of (6) is established in exactly the same way by considering the system (6) with $t \to -t$, i.e.,

$$\dot{\mathbf{y}} = -B\mathbf{y} - \mathbf{G}(\mathbf{y}).$$

The stable manifold for this system will then be the unstable manifold $\tilde{U}$ for (6). Note that it is also necessary to replace the vector y by the vector $(y_{k+1}, \ldots, y_n, y_1, \ldots, y_k)$ in order to determine the $n-k$ dimensional manifold U by the above process. This completes the proof of the Stable Manifold Theorem.

Remark 1. The first rigorous results concerning invariant manifolds were due to Hadamard in 1901, Liapunov in 1907 and Perron in 1928. They proved the existence of stable and unstable manifolds of systems of differential equations and of maps. The proof presented in this section is due to Liapunov and Perron. Several recent results generalizing the results of the Stable Manifold Theorem have been given by Hale, Hirsch, Pugh, Shub and Smale to mention a few. We note that if the function f $\in C^r(E)$ and $r \geq 1$, then the stable and unstable differentiable manifolds S and U of (1) are of class C^r. And if f is analytic in E then S and U are analytic manifolds.

Definition 3. Let ϕ_t be the flow of the nonlinear system (1). The global stable and unstable manifolds of (1) at 0 are defined by

$$W^s(0) = \bigcup_{t \leq 0} \phi_t(S)$$

and

$$W^u(0) = \bigcup_{t \geq 0} \phi_t(U)$$

respectively; $W^s(0)$ and $W^u(0)$ are also referred to as the global stable and unstable manifolds of the origin respectively. It can be shown that the global stable and unstable manifolds $W^s(0)$ and $W^u(0)$ are unique and

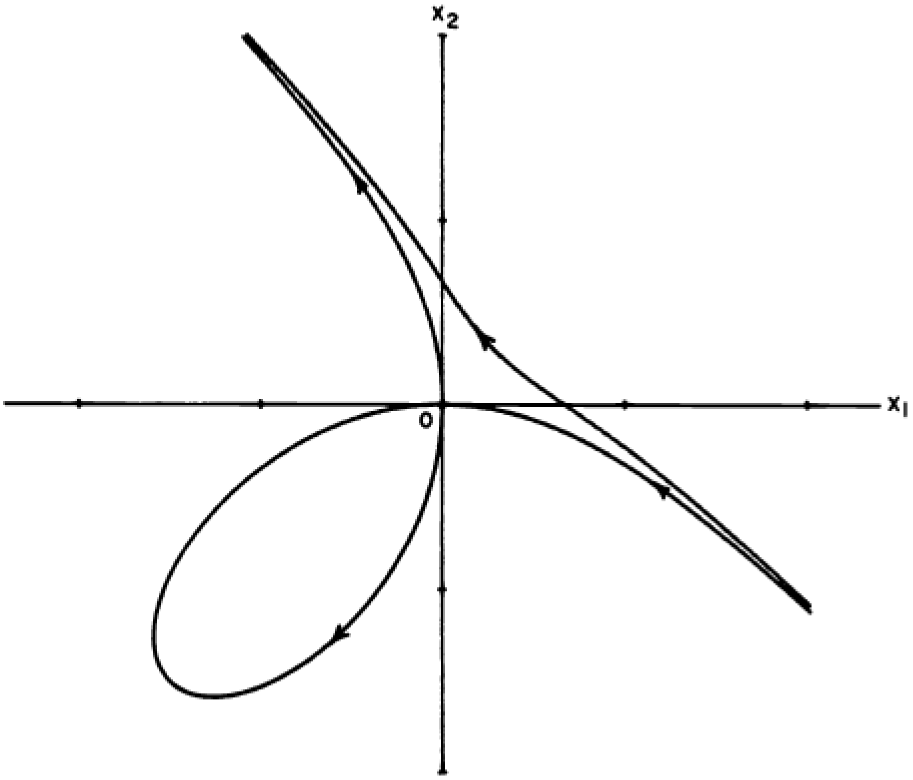

Figure 3

that they are invariant with respect to the flow ϕ_t; furthermore, for all $\mathbf{x} \in W^s(\mathbf{0})$, $\lim_{t\to\infty} \phi_t(\mathbf{x}) = \mathbf{0}$ and for all $\mathbf{x} \in W^u(0)$, $\lim_{t\to-\infty} \phi_t(\mathbf{x}) = \mathbf{0}$.

As in the proof of the Stable Manifold Theorem, it can be shown that in a small neighborhood, N, of a hyperbolic critical point at the origin, the local stable and unstable manifolds, S and U, of (1) at the origin are given by

$$S = \left\{\mathbf{x} \in N \mid \phi_t(\mathbf{x}) \to 0 \text{ as } t \to \infty \text{ and } \phi_t(\mathbf{x}) \in N \text{ for } t \geq 0\right\}$$

and

$$U = \left\{\mathbf{x} \in N \mid \phi_t(\mathbf{x}) \to 0 \text{ as } t \to -\infty \text{ and } \phi_t(\mathbf{x}) \in N \text{ for } t \leq 0\right\}$$

Figure 3 shows some numerically computed solution curves for the system in Example 2. The global stable and unstable manifolds for this example are shown in Figure 4. Note that $W^s(0)$ and $W^u(0)$ intersect in a "homoclinic loop" at the origin. $W^s(0)$ and $W^u(0)$ are more properly called "branched manifolds" in this example.

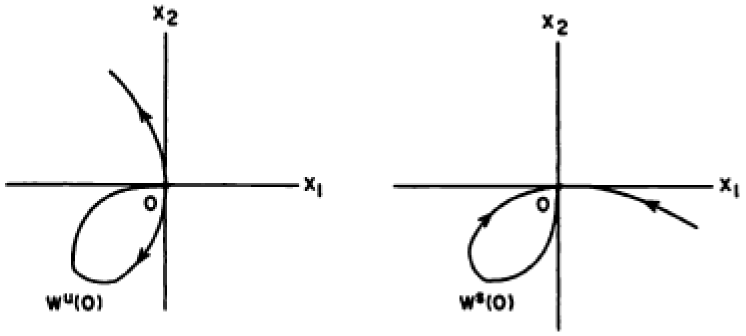

Figure 4

It follows from equation (11) in the proof of the stable manifold theorem that if $\mathbf{x}(t)$ is a solution of the differential equation (6) with $\mathbf{x}(0) \in S$, i.e., if $\mathbf{x}(t) = C\mathbf{y}(t)$ with $\mathbf{y}(0) = \mathbf{u}(0, \mathrm{a}) \in \tilde{S}$, then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x(0)| < \delta$ then

$$|\mathbf{x}(t)| \leq \varepsilon e^{-\alpha t}$$

for all $t \geq 0$. Just as in the proof of the stable manifold theorem, α is any positive number that satisfies $\operatorname{Re}\left(\lambda_j\right) < -\alpha$ for $j = 1, \ldots, k$ where λ_j, $j = 1, \ldots, k$ are the eigenvalues of $D\mathrm{f}(0)$ with negative real part. This result shows that solutions starting in S, sufficiently near the origin, approach the origin exponentially fast as $t \to \infty$.

Corollary. Under the hypotheses of the Stable Manifold Theorem, if S and U are the stable and unstable manifolds of (1) at the origin and if $\operatorname{Re}\left(\lambda_j\right) < -\alpha < 0 < \beta < \operatorname{Re}(\lambda_m)$ for $j = 1, \ldots, k$ and $m = k + 1, \ldots, n$, then given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\mathrm{x}_0 \in N_\delta(0) \cap S$ then

$$\left|\phi_t(\mathrm{x}_0)\right| \leq \varepsilon \mathrm{e}^{-\alpha t}$$

for all $t \geq 0$ and if $\mathrm{x}_0 \in N_\delta(0) \cap U$ then

$$\left|\phi_t(x_0)\right| \leq \varepsilon e^{\beta t}$$

for all $t \leq 0$. We add one final result to this section which establishes the existence of an invariant center manifold $W^c(0)$ tangent to E^c at 0. The next theorem follows from the local center manifold theorem, Theorem 2 in Section 2.12, and the stable manifold theorem in this section.

Theorem (The Center Manifold Theorem). Let $\mathbf{f} \in C^r(E)$ where E is an open subset of $\mathbf{R}^n$ containing the origin and $r \geq 1$. Suppose that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and that $D\mathbf{f}(\mathbf{0})$ has k eigenvalues with negative real part, j eigenvalues with positive real part, and $m = n - k - j$ eigenvalues with zero real part. Then there exists an m-dimensional center manifold $W^c(0)$ of class C^r tangent to the center subspace E^c of (2) at 0 , there exists a k-dimensional stable manifold $W^s(\mathbf{0})$ of class C^r tangent to the stable subspace E^s of (2) at 0 and there exists a j-dimensional unstable manifold $W^u(0)$ of class C^r tangent to the unstable subspace E^u of (2) at 0 ; furthermore, $W^c(0)$, $W^s(\mathbf{0})$ and $W^u(\mathbf{0})$ are invariant under the flow ϕ_t of (1).

1.7 The Hartman-Grobman Theorem

The Hartman-Grobman Theorem is another very important result in the local qualitative theory of ordinary differential equations. The theorem shows that near a hyperbolic equilibrium point x_0, the nonlinear system (1) has the same qualitative structure as the linear system (2), with $A = D\mathbf{f}(\mathbf{x}_0)$. Throughout this section we shall assume that the equilibrium point x_0 has been translated to the origin.

Definition 1. Two autonomous systems of differential equations such as (1) and (2) are said to be topologically equivalent in a neighborhood of the origin or to have the same qualitative structure near the origin if there is a homeomorphism H mapping an open set U containing the origin onto an open set V containing the origin which maps trajectories of (1) in U onto trajectories of (2) in V and preserves their orientation by time in the sense that if a trajectory is directed from $\mathbf{x}_1$ to $\mathbf{x}_2$ in U, then its image is directed from $H(\mathbf{x}_1)$ to $H(\mathbf{x}_2)$ in V. If the homeomorphism H preserves the parameterization by time, then the systems (1) and (2) are said to be topologically conjugate in a neighborhood of the origin.

Before stating the Hartman-Grobman Theorem, we consider a simple example of two topologically conjugate linear systems.

Theorem (The Hartman-Grobman Theorem). Let E be an open subset of $\mathbf{R}^n$ containing the origin, let $\mathrm{f} \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system (1). Suppose that $\mathrm{f}(\mathbf{0}) = \mathbf{0}$ and that the matrix $A = D\mathrm{f}(\mathbf{0})$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $\mathbf{x}_0 \in U$, there is an open interval $I_0 \subset \mathbf{R}$ containing zero such that for all $\mathrm{x}_0 \in U$ and $t \in I_0$

$$H \circ \phi_t(\mathbf{x}_0) = e^{At} H(\mathbf{x}_0);$$

i.e., H maps trajectories of (1) near the origin onto trajectories of (2) near the origin and preserves the parameterization by time.

Outline of the Proof. Consider the nonlinear system (1) with $\mathrm{f} \in C^1(E)$, $\mathrm{f}(\mathbf{0}) = \mathbf{0}$ and $A = D\mathrm{f}(\mathbf{0})$.

1. Suppose that the matrix A is written in the form

$$A = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

where the eigenvalues of P have negative real part and the eigenvalues of Q have positive real part.

2. Let ϕ_t be the flow of the nonlinear system (1) and write the solution

$$\mathbf{x}(t, \mathbf{x}_0) = \phi_t(\mathbf{x}_0) = \begin{bmatrix} \mathbf{y}(t, \mathbf{y}_0, \mathbf{z}_0) \\ \mathbf{z}(t, \mathbf{y}_0, \mathbf{z}_0) \end{bmatrix}$$

where

$$\mathbf{x}_0 = \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{z}_0 \end{bmatrix} \in \mathbf{R^n},$$

$\mathbf{y}_0 \in E^s$, the stable subspace of A and $z_0 \in E^u$, the unstable subspace of A.

3. Define the functions

$$\tilde{\mathbf{Y}}(\mathbf{y}_0, z_0) = \mathbf{y}(1, \mathbf{y}_0, \mathbf{z}_0) - e^P \mathbf{y}_0$$

and

$$\tilde{\mathbf{Z}}(\mathbf{y}_0, \mathbf{z}_0) = \mathbf{z}(1, \mathbf{y}_0, \mathbf{z}_0) - e^{Q z_0}.$$

Then $\tilde{\mathbf{Y}}(\mathbf{0}) = \tilde{\mathbf{Z}}(0) = D\tilde{\mathbf{Y}}(\mathbf{0}) = D\tilde{\mathbf{Z}}(\mathbf{0}) = 0$. And since $\mathbf{f} \in C^{\mathbf{1}}(E)$, $\tilde{\mathbf{Y}}(\mathbf{y}_0, \mathbf{z}_0)$ and $\tilde{\mathbf{Z}}(\mathbf{y}_0, \mathbf{z}_0)$ are continuously differentiable. Thus,

$$\left\| D\tilde{\mathbf{Y}}(\mathbf{y}_0, \mathbf{z}_0) \right\| \le a$$

and

$$\left\| D\tilde{\mathbf{Z}}(\mathbf{y}_0, \mathbf{z}_0) \right\| \le a$$

on the compact set $\left|y_0\right|^2 + |z_0|^2 \le s_0^2$. The constant a can be taken as small as we like by choosing s_0 sufficiently small. We let $\mathbf{Y}(\mathbf{y}_0, \mathbf{z}_0)$ and $\mathbf{Z}(\mathbf{y}_0, \mathbf{z}_0)$ be smooth functions which are equal to $\tilde{\mathbf{Y}}(\mathbf{y}_0, \mathbf{z}_0)$ and $\tilde{\mathbf{Z}}(\mathbf{y}_0, \mathbf{z}_0)$ for $\left|y_0\right|^2 + |z_0|^2 \le (s_0/2)^2$ and zero for $\left|y_0\right|^2 + |z_0|^2 \ge s_0^2$. Then by the mean value theorem

$$\left|\mathbf{Y}(\mathbf{y}_0, \mathbf{z}_0)\right| \le a\sqrt{\left|\mathbf{y}_0\right|^2 + |\mathbf{z}_0|^2} \le a\left(\left|\mathbf{y}_0\right| + |\mathbf{z}_0|\right)$$

and

$$\left|\mathbf{Z}(\mathbf{y}_0, \mathbf{z}_0)\right| \le a\sqrt{\left|\mathbf{y}_0\right|^2 + |z_0|^2} \le a\left(\left|\mathbf{y}_0\right| + |z_0|\right)$$

for all $(\mathbf{y}_0, \mathbf{z}_0) \in \mathbf{R^n}$. We next let $B = e^P$ and $C = e^Q$. Then assuming that we have carried out the normalization in Problem 7 in Section 1.8 of Chapter 1, cf. Hartman [H], p. 233, we have

$$b = \|B\| < 1 \text{ and } c = \left\|C^{-1}\right\| < 1.$$

4. For

$$\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} \in \mathbf{R}^n$$

define the transformations

$$L(\mathbf{y},\mathbf{z}) = \begin{bmatrix} B\mathbf{y} \\ C\mathbf{z} \end{bmatrix}$$

and

$$T(\mathbf{y},\mathrm{z}) = \begin{bmatrix} B\mathbf{y} + \mathbf{Y}(\mathbf{y},\mathrm{z}) \\ C\mathrm{z} + \mathbf{Z}(\mathbf{y},\mathrm{z}) \end{bmatrix}$$

i.e., $L(\mathbf{x}) = e^{A}\mathbf{x}$ and locally $T(\mathbf{x}) = \phi_1(\mathbf{x})$.

Lemma. There exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that

$$H \circ T = L \circ H.$$

We establish this lemma using the method of successive approximations. For $\mathbf{x} \in \mathbf{R}^{\mathbf{n}}$, let

$$H(\mathbf{x}) = \begin{bmatrix} \Phi(\mathbf{y},\mathbf{z}) \\ \Psi(\mathbf{y},\mathbf{z}) \end{bmatrix}$$

Then $H \circ T = L \circ H$ is equivalent to the pair of equations

$$\begin{aligned} B\Phi(\mathbf{y},\mathbf{z}) &= \Phi(B\mathbf{y} + \mathbf{Y}(\mathbf{y},\mathbf{z}), C\mathbf{z} + \mathbf{Z}(\mathbf{y},\mathbf{z})) \\ C\Psi(\mathbf{y},\mathbf{z}) &= \Psi(B\mathbf{y} + \mathbf{Y}(\mathbf{y},\mathbf{z}), C\mathbf{z} + \mathbf{Z}(\mathbf{y},\mathbf{z})). \end{aligned} \tag{3}$$

First of all, define the successive approximations for the second equation by

$$\begin{aligned} \Psi_0(\mathbf{y},\mathbf{z}) &= \mathbf{z} \\ \Psi_{k+1}(\mathbf{y},\mathbf{z}) &= C^{-1}\Psi_k(B\mathbf{y} + \mathbf{Y}(\mathbf{y},\mathbf{z}), C\mathbf{z} + \mathbf{Z}(\mathbf{y},\mathbf{z})) \end{aligned} \tag{4}$$

It then follows by an easy induction argument that for $k = 0, 1, 2, \ldots$, the $\Psi_k(\mathbf{y},\mathbf{z})$ are continuous and satisfy $\Psi_k(\mathbf{y},\mathbf{z}) = \mathbf{z}$ for $|\mathbf{y}| + |\mathbf{z}| \geq 2s_0$. We next prove by induction that for $j = 1, 2, \ldots$

$$\left|\Psi_j(\mathbf{y},\mathbf{z}) - \Psi_{j-1}(\mathbf{y},\mathbf{z})\right| \leq Mr^j(|\mathbf{y}| + |\mathbf{z}|)^\delta$$

where $r = \mathrm{c}[2\max(a,b,\mathrm{c})]^\delta$ with $\delta \in (0,1)$ chosen sufficiently small so that $r < 1$ (which is possible since $c < 1$) and $M = ac\,(2s_0)^{1-\delta}/r$. First of all for $j = 1$

$$\begin{aligned} \left|\Psi_1(\mathbf{y},\mathbf{z}) - \Psi_0(\mathbf{y},\mathbf{z})\right| &= C^{-1}\Psi_0(B\mathbf{y} + \mathbf{Y}(\mathbf{y},\mathbf{z}), C\mathbf{z} + \mathbf{Z}(\mathbf{y},\mathbf{z})) - \mathbf{z}\,| \\ &= \left|C^{-1}(C\mathbf{z} + \mathbf{Z}(\mathbf{y},\mathbf{z})) - \mathbf{z}\right| \\ &= \left|C^{-1}\mathbf{Z}(\mathbf{y},\mathbf{z})\right| \leq \left\|C^{-1}\right\| |\mathbf{Z}(\mathbf{y},\mathbf{z})| \\ &\leq ca(|\mathbf{y}| + |\mathbf{z}|) \leq Mr(|\mathbf{y}| + |\mathbf{z}|)^\delta \end{aligned}$$

since $\mathbf{Z}(\mathbf{y},\mathbf{z}) = 0$ for $|\mathbf{y}| + |\mathbf{z}| \geq 2s_0$. And then assuming that the induction hypothesis holds for $j = 1, \ldots, k$ we have

$$\begin{aligned}
\left|\Psi_{k+1}(\mathbf{y},\mathbf{z}) - \Psi_k(\mathbf{y},\mathbf{z})\right| &= |\; C^{-1}\Psi_k(B\mathbf{y} + \mathbf{Y}(\mathbf{y},\mathbf{z}), C\mathbf{z} + \mathbf{Z}(\mathbf{y},\mathbf{z})) \\
&\quad - C^{-1}\Psi_{k-1}(B\mathbf{y} + \mathbf{Y}(\mathbf{y},\mathbf{z}), C\mathbf{z} + \mathbf{Z}(\mathbf{y},\mathbf{z}))\;| \\
&\leq \left\|C^{-1}\right\| \;|\; \Psi_k('') - \Psi_{k-1}('')\;| \\
&\leq cMr^k \left[|B\mathbf{y} + \mathbf{Y}(\mathbf{y},\mathbf{z})| + |C\mathbf{z} + \mathbf{Z}(\mathbf{y},\mathbf{z})\right]^\delta \\
&\leq cMr^k |b|\mathbf{y}| + 2a(|\mathbf{y}| + |\mathrm{z}|) + c|\mathbf{z}\;|\Big]^\delta \\
&\leq cMr^k [2\max(a,b,c)]^\delta (|\mathbf{y}| + |\mathrm{z}|)^\delta \\
&= Mr^{k+1}(|\mathbf{y}| + |\mathrm{z}|)^\delta
\end{aligned}$$

Thus, just as in the proof of the fundamental theorem in Section 2.2 and the stable manifold theorem in Section 2.7, $\Psi_k(y,z)$ is a Cauchy sequence of continuous functions which converges uniformly as $k \to \infty$ to a continuous function $\Psi(y,z)$. Also, $\Psi(y,z) = z$ for $|y| + |z| \geq 2s_0$. Taking limits in (4) shows that $\Psi(\mathbf{y},\mathbf{z})$ is a solution of the second equation in (3).

The first equation in (3) can be written as

$$B^{-1}\Phi(y,z) = \Phi\left(B^{-1}y + Y_1(y,z), C^{-1}z + Z_1(y,z)\right)$$

where the functions Y_1 and Z_1 are defined by the inverse of T (which exists if the constant a is sufficiently small, i.e., if s_0 is sufficiently small) as follows:

$$T^{-1}(y,z) = \begin{bmatrix} B^{-1}y + Y_1(y,z) \\ C^{-1}z + Z_1(\mathbf{y},z) \end{bmatrix}.$$

Then equation (6) can be solved for $\Phi(y,z)$ by the method of successive approximations exactly as above with $\Phi_0(\mathbf{y},\mathbf{z}) = \mathbf{y}$ since $b = \|B\| < 1$. We therefore obtain the continuous map

$$H(\mathbf{y},\mathrm{z}) = \begin{bmatrix} \Phi(\mathbf{y},\mathrm{z}) \\ \Psi(\mathbf{y},\mathrm{z}) \end{bmatrix}$$

And it follows as on pp. 248-249 in Hartman [H] that H is a homeomorphism of $\mathbf{R}^\mathbf{n}$ onto $\mathbf{R}^n$.

5. We now let H_0 be the homeomorphism defined above and let L^t and T^t be the one-parameter families of transformations defined by

$$L^t\,(\mathrm{x}_0) = \mathrm{e}^{At}\mathrm{x}_0 \quad \text{and} \quad T^t\,(\mathrm{x}_0) = \phi_t\,(\mathrm{x}_0)\,.$$

Define

$$H = \int_0^1 L^{-s} H_0 T^s ds.$$

It then follows using the above lemma that there exists a neighborhood of the origin for which

$$\begin{aligned} L^t H &= \int_0^1 L^{t-s} H_0 T^{s-t} ds T^t \\ &= \int_{-t}^{1-t} L^{-s} H_0 T^s ds T^t \\ &= \left[\int_{-t}^{0} L^{-s} H_0 T^s ds + \int_0^{1-t} L^{-s} H_0 T^s ds \right] T^t \\ &= \int_0^1 L^{-s} H_0 T^s ds T^t = H T^t \end{aligned}$$

since by the above lemma $H_0 = L^{-1} H_0 T$ which implies that

$$\begin{aligned} \int_{-t}^{0} L^{-s} H_0 T^s ds &= \int_{-t}^{0} L^{-s-1} H_0 T^{s+1} ds \\ &= \int_{1-t}^{1} L^{-s} H_0 T^s ds \end{aligned}$$

Thus, $H \circ T^t = L^t H$ or equivalently

$$H \circ \phi_t (\mathrm{x}_0) = \mathrm{e}^{At} H (\mathrm{x}_0)$$

and it can be shown as on pp. 250-251 of Hartman [H] that H is a homeomorphism on $\mathbf{R}^n$. This completes the outline of the proof of the HartmanGrobman Theorem.

Theorem (Hartman). Let E be an open subset of $\mathbf{R}^\mathbf{n}$ containing the point $\mathbf{x}_0$, let $\mathbf{f} \in C^2(E)$, and let ϕ_t be the flow of the nonlinear system (1). Suppose that $\mathbf{f}(\mathbf{x}_0) = 0$ and that all of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $A = D\mathbf{f}(\mathbf{x}_0)$ have negative (or positive) real part. Then there exists a C^1-diffeomorphism H of a neighborhood U of $\mathbf{x}_0$ onto an open set V containing the origin such that for each $\mathbf{x} \in U$ there is an open interval $I(\mathbf{x}) \subset \mathbf{R}$ containing zero such that for all $\mathbf{x} \in U$ and $t \in I(\mathbf{x})$

$$H \circ \phi_t(\mathbf{x}) = e^{At} H(\mathbf{x}).$$

1.8 Stability and Liapunov Functions

In this section we discuss the stability of the equilibrium points of the nonlinear system (1). The stability of any hyperbolic equilibrium point $\mathbf{x}_0$ of (1) is determined by the signs of the real parts of the eigenvalues λ_j of the matrix $Df(\mathrm{x}_0)$. A hyperbolic equilibrium point $\mathbf{x}_0$ is asymptotically stable iff $\mathrm{Re}\left(\lambda_j\right) < 0$ for

$j = 1, \ldots, n$; i.e., iff x_0 is a sink. And a hyperbolic equilibrium point x_0 is unstable iff it is either a source or a saddle. The stability of nonhyperbolic equilibrium points is typically more difficult to determine. A method, due to Liapunov, that is very useful for deciding the stability of nonhyperbolic equilibrium points is presented in this section.

Definition 1. Let ϕ_t denote the flow of the differential equation (1) defined for all $t \in \mathbf{R}$. An equilibrium point $\mathbf{x}_0$ of (1) is stable if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $\mathbf{x} \in N_6(\mathbf{x}_0)$ and $t \geq 0$ we have

$$\phi_t(\mathrm{x}) \in N_c(\mathrm{x}_0) .$$

The equilibrium point $\mathbf{x}_0$ is unstable if it is not stable. And $\mathbf{x}_0$ is asymptotically stable if it is stable and if there exists a $\delta > 0$ such that for all $\mathbf{x} \in N_6(\mathbf{x}_0)$ we have

$$\lim_{t \to \infty} \phi_t(\mathbf{x}) = \mathrm{x}_0.$$

Note that the above limit being satisfied for all $\mathbf{x}$ in some neighborhood of $\mathbf{x}_0$ does not imply that $\mathbf{x}_0$ is stable.

It can be seen from the phase portraits in Section 1.5 of Chapter 1 that a stable node or focus of a linear system in $\mathbf{R}^2$ is an asymptotically stable equilibrium point; an unstable node or focus or a saddle of a linear system in $\mathbf{R}^2$ is an unstable equilibrium point; and a center of a linear system in $\mathbf{R}^2$ is a stable equilibrium point which is not asymptotically stable.

It follows from the Stable Manifold Theorem and the Hartman-Grobman Theorem that any sink of (1) is asymptotically stable and any source or saddle of (1) is unstable. Hence, any hyperbolic equilibrium point of (1) is either asymptotically stable or unstable. The corollary in Section 2.7 provides even more information concerning the local behavior of solutions near a sink:

Theorem 1. If x_0 is a sink of the nonlinear system (1) and $\mathrm{Re}\left(\lambda_j\right) < -\alpha < 0$ for all of the eigenvalues λ_j of the matrix $\mathrm{Df}(\mathrm{x}_0)$, then given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $\mathbf{x} \in N_\delta(\mathbf{x}_0)$, the flow $\phi_t(\mathbf{x})$ of (1) satisfies

$$\left|\phi_t(\mathbf{x}) - \mathbf{x}_0\right| \leq \varepsilon e^{-\alpha t}$$

for all $t \geq 0$.

Since hyperbolic equilibrium points are either asymptotically stable or unstable, the only time that an equilibrium point x_0 of (1) can be stable but not asymptotically stable is when $D\mathbf{f}(\mathbf{x}_0)$ has a zero eigenvalue or a pair of complex-conjugate, pure-imaginary eigenvalues $\lambda = \pm ib$. It follows from the next theorem, proved in [H/S], that all other eigenvalues λ, of $\mathrm{Df}(\mathrm{x}_0)$ must satisfy $\mathrm{Re}\left(\lambda_j\right) \leq 0$ if x_0 is stable.

Theorem 2. If x_0 is a stable equilibrium point of (1), no eigenvalue of $\mathrm{Df}(\mathrm{x}_0)$ has positive real part.

We see that stable equilibrium points which are not asymptotically stable can only occur at nonhyperbolic equilibrium points. But the question as to whether a nonhyperbolic equilibrium point is stable, asymptotically stable or unstable is a delicate question.

The following method, due to Liapunov (in his 1892 doctoral thesis), is very useful in answering this question.

Definition 2. If $\mathbf{f} \in C^1(E)$, $V \in C^1(E)$ and ϕ_t is the flow of the differential equation (1), then for $\mathbf{x} \in E$ the derivative of the function $V(\mathbf{x})$ along the solution $\phi_t(\mathbf{x})$

$$\dot{V}(\mathbf{x}) = \frac{d}{dt} V\left(\phi_t(\mathbf{x})\right)\Big|_{t=0} = DV(\mathbf{x})\mathbf{f}(\mathbf{x}).$$

The last equality follows from the chain rule. If $\dot{V}(\mathbf{x})$ is negative in E then $V(\mathbf{x})$ decreases along the solution $\phi_t(\mathrm{x}_0)$ through $\mathrm{x}_0 \in E$ at $t = 0$. Furthermore, in $\mathbf{R}^2$, if $\dot{V}(\mathbf{x}) \leq 0$ with equality only at $\mathbf{x} = 0$, then for small positive C, the family of curves $V(\mathbf{x}) = C$ constitutes a family of closed curves enclosing the origin and the trajectories of (1) cross these curves from their exterior to their interior with increasing t; i.e., the origin of (1) is asymptotically stable. A function $V : \mathbf{R^n} \to \mathbf{R}$ satisfying the hypotheses of the next theorem is called a Liapunov function.

Theorem 3. Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$. Suppose that $\mathrm{f} \in C^1(E)$ and that $\mathrm{f}(\mathrm{x}_0) = \mathbf{0}$. Suppose further that there exists a real valued function $V \in C^1(E)$ satisfying $V(\mathrm{x}_0) = 0$ and $V(\mathbf{x}) > 0$ if $\mathbf{x} \neq \mathrm{x}_0$.
Then (a) if $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in E$, $\mathbf{x}_0$ is stable;
(b) if $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in E \sim \{\mathbf{x}_0\}$, $\mathbf{x}_0$ is asymptotically stable;
(c) if $\dot{V}(\mathbf{x}) > 0$ for all $\mathbf{x} \in E \sim \{\mathbf{x}_0\}$, $\mathbf{x}_0$ is unstable.

Proof. Without loss of generality, we shall assume that the equilibrium point $\mathbf{x}_0 = \mathbf{0}$.

(a) Choose $\varepsilon > 0$ sufficiently small that $\overline{N_\varepsilon(0)} \subset E$ and let m_ε be the minimum of the continuous function $V(\mathbf{x})$ on the compact set

$$S_\varepsilon = \{\mathbf{x} \in \mathbf{R^n} | \quad |\mathbf{x}| = \varepsilon\}.$$

Then since $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$, it follows that $m_\varepsilon > 0$. Since $V(\mathbf{x})$ is continuous and $V(0) = 0$, it follows that there exists a $\delta > 0$ such that $|\mathbf{x}| < \delta$ implies that $V(\mathbf{x}) < m_c$. Since $\dot{V}(\mathbf{x}) \leq 0$ for $\mathbf{x} \in E$, it follows that $V(\mathbf{x})$ is decreasing along trajectories of (1). Thus, if ϕ_t is the flow of the differential equation (1), it follows that for all $\mathrm{x}_0 \in N_\delta(0)$ and $t \geq 0$ we have

$$V\left(\phi_t(\mathrm{x}_0)\right) \leq V(\mathrm{x}_0) < m_\varepsilon.$$

Now suppose that for $|\mathbf{x}_0| < \delta$ there is a $t_1 > 0$ such that $\left|\phi_{t_1}(\mathbf{x}_0)\right| = \varepsilon$; i.e., such that $\phi_{t_1}(\mathrm{x}_0) \in S_\varepsilon$. Then since m_ε is the minimum of $V(\mathbf{x})$ on S_ε, this would imply that

$$V\left(\phi_{t_1}(\mathrm{x}_0)\right) \geq m_\varepsilon$$

which contradicts the above inequality. Thus for $|x_0| < \delta$ and $t \geq 0$ it follows that $\left|\phi_t(\mathrm{x}_0)\right| < \varepsilon$; i.e., 0 is a stable equilibrium point.

(b) Suppose that $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in E$. Then $V(\mathbf{x})$ is strictly decreasing along trajectories of (1). Let ϕ_t be the flow of (1) and let $\mathrm{x}_0 \in N_\delta(0)$, the neighborhood defined in part (a). Then, by part (a), if $|x_0| < \delta$, $\phi_t(\mathrm{x}_0) \subset N_\varepsilon(0)$ for all $t \geq 0$. Let $\{t_k\}$ be any sequence with $t_k \to \infty$. Then since $\overline{N_\varepsilon(0)}$ is compact, there is a subsequence of $\left\{\phi_{t_k}(\mathrm{x}_0)\right\}$ that converges to a point in $\overline{N_\varepsilon(\mathbf{0})}$. But for any subsequence $\{t_n\}$ of $\{t_k\}$ such that $\left\{\phi_{t_n}(\mathbf{x}_0)\right\}$ converges, we show below that the limit is zero. It then follows that $\phi_{t_k}(\mathrm{x}_0) \to 0$ for any sequence $t_k \to \infty$ and therefore that $\phi_t(\mathrm{x}_0) \to 0$ as $t \to \infty$; i.e., that 0 is asymptotically stable. It remains to show that if $\phi_{t_n}(x_0) \to y_0$, then $y_0 = 0$. Since $V(x)$ is strictly decreasing along trajectories of (1) and since $V\left(\phi_{t_n}(\mathbf{x}_0)\right) \to V(\mathbf{y}_0)$ by the continuity of V, it follows that

$$V\left(\phi_t(\mathbf{x}_0)\right) > V(\mathbf{y}_0)$$

for all $t > 0$. But if $\mathbf{y}_0 \neq 0$, then for $s > 0$ we have $V\left(\phi_s(\mathbf{y}_0)\right) < V(\mathbf{y}_0)$ and, by continuity, it follows that for all y sufficiently close to y_0 we have $V\left(\phi_s(\mathbf{y})\right) < V(\mathbf{y}_0)$ for $s > 0$. But then for $\mathbf{y} = \phi_{t_n}(\mathrm{x}_0)$ and n sufficiently large, we have

$$V\left(\phi_{s+t_n}(\mathrm{x}_0)\right) < V(\mathbf{y}_0)$$

which contradicts the above inequality. Therefore $\mathbf{y}_0 = \mathbf{0}$ and it follows that 0 is asymptotically stable.

(c) Let M be the maximum of the continuous function $V(\mathbf{x})$ on the compact set $\overline{N_\epsilon(\mathbf{0})}$. Since $\dot{V}(\mathbf{x}) > 0$, $V(\mathbf{x})$ is strictly increasing along trajectories of (1). Thus, if ϕ_t is the flow of (1), then for any $\delta > 0$ and $\mathbf{x}_0 \in N_\delta(\mathbf{0}) \sim \{0\}$ we have

$$V\left(\phi_t(\mathbf{x}_0)\right) > V(\mathrm{x}_0) > 0$$

for all $t > 0$. And since $\dot{V}(\mathbf{x})$ is positive definite, this last statement implies that

$$\inf_{i\geq 0} \dot{V}\left(\phi_t(\mathbf{x}_0)\right) = m > 0$$

Thus,

$$V\left(\phi_t(\mathbf{x}_0)\right) - V(\mathbf{x}_0) \geq mt$$

for all $t \geq 0$. Therefore,

$$V\left(\phi_t\left(\mathbf{x}_0\right)\right) > mt > M$$

for t sufficiently large; i.e., $\phi_t\left(\mathbf{x}_0\right)$ lies outside the closed set $\overline{N_\varepsilon(\mathbf{0})}$. Hence, $\mathbf{0}$ is unstable.

Remark. If $\dot{V}(\mathbf{x}) = 0$ for all $\mathbf{x} \in E$ then the trajectories of (1) lie on the surfaces in $\mathbf{R}^n$ (or curves in $\mathbf{R}^2$) defined by

$$V(\mathbf{x}) = \mathbf{c}.$$

1.9 Saddles, Nodes, Foci and Centers

In Section 1.5 of Chapter 1, a linear system (1), where $\mathbf{x} \in \mathbf{R}^2$ was said to have a saddle, node, focus or center at the origin if its phase portrait was linearly equivalent to one of the phase portraits in Figures 1-4 in Section 1.5 of Chapter 1 respectively; i.e., if there exists a nonsingular linear transformation which reduces the matrix A to one of the canonical matrices B in Cases I-IV of Section 1.5 in Chapter 1 respectively. For example, the linear system (1) of the example in Section 2.8 of this chapter has a saddle at the origin.

In Section 2.6, a nonlinear system (2) was said to have a saddle, a sink or a source at a hyperbolic equilibrium point x_0 if the linear part of f at x_0 had eigenvalues with both positive and negative real parts, only had eigenvalues with negative real parts, or only had eigenvalues with positive real parts, respectively.

In this section, we define the concept of a topological saddle for the nonlinear system (2) with $x \in R^2$ and show that if x_0 is a hyperbolic equilibrium point of (2) then it is a topological saddle if and only if it is a saddle of (2); i.e., a hyperbolic equilibrium point x_0 is a topological saddle for (2) if and only if the origin is a saddle for (1) with $A = D\mathbf{f}\left(\mathbf{x}_0\right)$. We discuss topological saddles for nonhyperbolic equilibrium points of (2) with $\mathbf{x} \in \mathbf{R}^2$ in the next section. We also refine the classification of sinks of the nonlinear system (2) into stable nodes and foci and show that, under slightly stronger hypotheses on the function f, i.e., stronger than $\mathrm{f} \in C^1(E)$, a hyperbolic critical point x_0 is a stable node or focus for the nonlinear system (2) if and only if it is respectively a stable node or focus for the linear system (1) with $A = D\mathrm{f}(\mathrm{x}_0$). Similarly, a source of (2) is either an unstable node or focus of (2) as defined below. Finally, we define centers and center-foci for the nonlinear system (2) and show that, under the addition of nonlinear terms, a center of the linear system (1) may become either a center, a center-focus, or a stable or unstable focus of (2).

Before defining these various types of equilibrium points for planar systems (2), it is convenient to introduce polar coordinates (r, θ) and to rewrite the system (2) in polar coordinates. In this section we let $\mathbf{x} = (x, y)^T$, $f_1(\mathbf{x}) = P(x, y)$ and $f_2(\mathbf{x}) = Q(x, y)$. The nonlinear system (2) can then be written as

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y). \tag{3}$$

If we let $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$, then we have

$$r\dot{r} = x\dot{x} + y\dot{y}$$

and

$$r^2\dot{\theta} = x\dot{y} - y\dot{x}.$$

It follows that for $r > 0$, the nonlinear system (3) can be written in terms of polar coordinates as

$$\dot{r} = P(r\cos\theta, r\sin\theta)\cos\theta + Q(r\cos\theta, r\sin\theta)\sin\theta$$
$$r\dot{\theta} = Q(r\cos\theta, r\sin\theta)\cos\theta - P(r\cos\theta, r\sin\theta)\sin\theta \tag{4}$$

or as

$$\frac{dr}{d\theta} = F(r, \theta) \equiv \frac{r[P(r\cos\theta, r\sin\theta)\cos\theta + Q(r\cos\theta, r\sin\theta)\sin\theta]}{Q(r\cos\theta, r\sin\theta)\cos\theta - P(r\cos\theta, r\sin\theta)\sin\theta}. \tag{5}$$

Writing the system of differential equations (3) in polar coordinates will often reveal the nature of the equilibrium point or critical point at the origin.

Definition 1. The origin is called a center for the nonlinear system (2) if there exists a $\delta > 0$ such that every solution curve of (2) in the deleted neighborhood $N_\delta(0) \sim \{0\}$ is a closed curve with 0 in its interior.

Definition 2. The origin is called a center-focus for (2) if there exists a sequence of closed solution curves Γ_n with Γ_{n+1} in the interior of Γ_n such that $\Gamma_n \to 0$ as $n \to \infty$ and such that every trajectory between Γ_n and Γ_{n+1} spirals toward Γ_n or Γ_{n+1} as $t \to \pm\infty$.

Definition 3. The origin is called a stable focus for (2) if there exists a $\delta > 0$ such that for $0 < r_0 < \delta$ and $\theta_0 \in \mathbf{R}$, $r(t, r_0, \theta_0) \to 0$ and $|\theta(t, r_0, \theta_0)| \to \infty$ as $t \to \infty$. It is called an unstable focus if $r(t, r_0, \theta_0) \to 0$ and $|\theta(t, r_0, \theta_0)| \to \infty$ as $t \to -\infty$. Any trajectory of (2) which satisfies $r(t) \to 0$ and $|\theta(t)| \to \infty$ as $t \to \pm\infty$ is said to spiral toward the origin as $t \to \pm\infty$.

Definition 4. The origin is called a stable node for (2) if there exists a $\delta > 0$ such that for $0 < r_0 < \delta$ and $\theta_0 \in \mathbf{R}$, $r(t, r_0, \theta_0) \to 0$ as $t \to \infty$ and $\lim_{t\to\infty} \theta(t, r_0, \theta_0)$ exists; i.e., each trajectory in a deleted neighborhood of the origin approaches the origin along a well-defined tangent line as $t \to \infty$. The origin is called an unstable node if $r(t, r_0, \theta_0) \to 0$ as $t \to -\infty$ and $\lim_{t\to-\infty} \theta(t, r_0, \theta_0)$ exists for all $r_0 \in (0, \delta)$ and $\theta_0 \in \mathbf{R}$. The origin is called a proper node for (2) if it is a node and if every ray through the origin is tangent to some trajectory of (2).

Definition 5. The origin is a (topological) saddle for (2) if there exist two trajectories Γ_1 and Γ_2 which approach 0 as $t \to \infty$ and two trajectories Γ_3 and Γ_4 which approach 0 as $t \to -\infty$ and if there exists a $\delta > 0$ such that all other trajectories which start in the deleted neighborhood of the origin $N_6(\mathbf{0}) \sim \{0\}$ leave $N_6(\mathbf{0})$ as $t \to \pm\infty$. The special trajectories $\Gamma_1, \ldots, \Gamma_4$ are called separatrices.

For a (topological) saddle, the stable manifold at the origin $S = \Gamma_1 \cup \Gamma_2 \cup \{0\}$ and the unstable manifold at the origin $U = \Gamma_3 \cup \Gamma_4 \cup \{0\}$. If the trajectory Γ_i approaches the origin along a ray making an angle θ_i with the x-axis where $\theta_i \in (-\pi, \pi]$ for $i = 1, \ldots, 4$, then $\theta_2 = \theta_1 \pm \pi$ and $\theta_4 = \theta_3 \pm \pi$. This follows by considering the possible directions in which a trajectory of (2), written in polar form (4), can approach the origin; cf. equation (6) below. The following theorems, proved in [A-I], are useful in this regard. The first theorem is due to Bendixson [B].

Theorem 1 (Bendixson). Let E be an open subset of $\mathbf{R}^2$ containing the origin and let f $\in C^1(E)$. If the origin is an isolated critical point of (2), then either every neighborhood of the origin contains a closed solution curve with 0 in its interior or there exists a trajectory approaching 0 as $t \to \pm\infty$.

Theorem 2. Suppose that $P(x, y)$ and $Q(x, y)$ in (3) are analytic functions of x and y in some open subset E of $\mathbf{R}^2$ containing the origin and suppose that the Taylor expansions of P and Q about $(0, 0)$ begin with mth-degree terms $P_m(x, y)$ and $Q_m(x, y)$ with $m \geq 1$. Then any trajectory of (3) which approaches the origin as $t \to \infty$ either spirals toward the origin as $t \to \infty$ or it tends toward the origin in a definite direction $\theta = \theta_0$ as $t \to \infty$. If $xQ_m(x, y) - yP_m(x, y)$ is not identically zero, then all directions of approach, θ_0, satisfy the equation

$$\cos\theta_0 Q_m(\cos\theta_0, \sin\theta_0) - \sin\theta_0 P_m(\cos\theta_0, \sin\theta_0) = 0.$$

Furthermore, if one trajectory of (3) spirals toward the origin as $t \to \infty$ then all trajectories of (3) in a deleted neighborhood of the origin spiral toward 0 as $t \to \infty$.

It follows from this theorem that if P and Q begin with first-degree terms, i.e., if

$$P_1(x, y) = ax + by$$

and

$$Q_1(x, y) = cx + dy$$

with $a, b,$ c and d not all zero, then the only possible directions in which trajectories can approach the origin are given by directions θ which satisfy

$$b\sin^2\theta + (a-d)\sin\theta\cos\theta - \cos^2\theta = 0. \tag{6}$$

For $\cos\theta \neq 0$ in this equation, i.e., if $b \neq 0$, this equation is equivalent to

$$b\tan^2\theta + (a-d)\tan\theta - \text{c} = 0. \tag{6'}$$

This quadratic has at most two solutions $\theta \in (-\pi/2, \pi/2]$ and if $\theta = \theta_1$ is a solution then $\theta = \theta_1 \pm \pi$ are also solutions. Finding the solutions of (6′) is equivalent to finding the directions determined by the eigenvectors of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The next theorem follows immediately from the Stable Manifold Theorem and the Hartman-Grobman Theorem. It establishes that if the origin is a hyperbolic equilibrium point of the nonlinear system (2), then it is a (topological) saddle for (2) if and only if it is a saddle for its linearization at the origin. Furthermore, the directions θ_j along which the separatrices Γ_j approach the origin are solutions of (6).

Theorem 3. Suppose that E is an open subset of $\mathbf{R}^2$ containing the origin and that $\text{f} \in C^1(E)$. If the origin is a hyperbolic equilibrium point of the nonlinear system (2), then the origin is a (topological) saddle for (2) if and only if the origin is a saddle for the linear system (1) with $A = D\text{f}(0)$.

Theorem 4. Let E be an open subset of $\mathbf{R}^2$ containing the origin and let $\mathbf{f} \in C^2(E)$. Suppose that the origin is a hyperbolic critical point of (2). Then the origin is a stable (or unstable) node for the nonlinear system (2) if and only if it is a stable (or unstble) node for the linear system (1) with $A = D\mathbf{f}(\mathbf{0})$. And the origin is a stable (or unstable) focus for the nonlinear system (2) if and only if it is a stable (or unstable) focus for the linear system (1) with $A = D\text{f}(\mathbf{0})$.

Remark. Under the hypotheses of Theorem 4, it follows that the origin is a proper node for the nonlinear system (2) if and only if it is a proper node for the linear system (1) with $A = D\text{f}(0)$. And under the weaker hypothesis that $\text{f} \in C^1(E)$, it still follows that if the origin is a focus for the linear system (1) with $A = D\mathbf{f}(0)$, then it is a focus for the nonlinear system (2).

Theorem 5. Let E be an open subset of $\mathbf{R}^2$ containing the origin and let $\mathbf{f} \in C^1(E)$ with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Suppose that the origin is a center for the linear system (1) with $A = D\mathbf{f}(\mathbf{0})$. Then the origin is either a center, a center-focus or a focus for the nonlinear system (2).

Proof. We may assume that the matrix $A = D\mathbf{f}(0)$ has been transformed to its canonical form

$$A = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$$

with $b \neq 0$. Assume that $b > 0$; otherwise we can apply the linear transformation $t \to -t$. The nonlinear system (3) then has the form

$$\dot{x} = -by + p(x, y)$$
$$\dot{y} = bx + q(x, y)$$

Since f $\in C^1(E)$, it follows that $|p(x, y)/r| \to 0$ and $|q(x, y)/r| \to 0$ as $r \to 0$; i.e., $p = o(r)$ and $q = o(r)$ as $r \to 0$. Thus, in polar coordinates we have $\dot{r} = \mathrm{o}(r)$ and $\dot{\theta} = b + o(1)$ as $r \to 0$. Therefore, there exists a $\delta > 0$ such that $\dot{\theta} \geq b/2 > 0$ for $0 < r \leq \delta$. Thus for $0 < r_0 \leq \delta$ and $\theta_0 \in \mathbf{R}$, $\theta\,(t, r_0, \theta_0) \geq bt/2 + \theta_0 \to \infty$ as $t \to \infty$; and $\theta\,(t, r_0, \theta_0)$ is a monotone increasing function of t. Let $t = h(\theta)$ be the inverse of this monotone function. Define $\tilde{r}(\theta) = r\,(h(\theta), r_0, \theta_0)$ for $0 < r_0 \leq \delta$ and $\theta_0 \in \mathbf{R}$. Then $\tilde{r}(\theta)$ satisfies the differential equation (5) which has the form

$$\frac{d\tilde{r}}{d\theta} = \tilde{F}(\tilde{r}, \theta) = \frac{\cos\theta p(\tilde{r}\cos\theta, \tilde{r}\sin\theta) + \sin\theta q(\tilde{r}\cos\theta, \tilde{r}\sin\theta)}{b + (\cos\theta/\tilde{r})q(\tilde{r}\cos\theta, \tilde{r}\sin\theta) - (\sin\theta/\tilde{r})p(\tilde{r}\cos\theta, \tilde{r}\sin\theta)}$$

Suppose that the origin is not a center or a center-focus for the nonlinear system (3). Then for $\delta > 0$ sufficiently small, there are no closed trajectories of (3) in the deleted neighborhood $N_\delta(0) \sim \{0\}$. Thus for $0 < r_0 < \delta$ and $\theta_0 \in \mathbf{R}$, either $\tilde{r}\,(\theta_0 + 2\pi) < \tilde{r}\,(\theta_0)$ or $\tilde{r}\,(\theta_0 + 2\pi) > \tilde{r}\,(\theta_0)$. Assume that the first case holds. The second case is treated in a similar manner. If $\tilde{r}\,(\theta_0 + 2\pi) < \tilde{r}\,(\theta_0)$ then $\tilde{r}\,(\theta_0 + 2k\pi) < \tilde{r}\,(\theta_0 + 2(k-1)\pi)$ for $k = 1, 2, 3\ldots$ Otherwise we would have two trajectories of (3) through the same point which is impossible. The sequence $\tilde{r}\,(\theta_0 + 2k\pi)$ is monotone decreasing and bounded below by zero; therefore, the following limit exists and is nonnegative:

$$\tilde{r}_1 = \lim_{k\to\infty} \tilde{r}\,(\theta_0 + 2k\pi)$$

If $\tilde{r}_1 = 0$ then $\tilde{r}(\theta) \to 0$ as $\theta \to \infty$; i.e., $r\,(t, r_0, \theta_0) \to 0$ and $\theta\,(t, r_0, \theta_0) \to \infty$ as $t \to \infty$ and the origin is a stable focus of (3). If $\tilde{r}_1 > 0$ then since $|\tilde{F}(r, \theta)| \leq M$ for $0 \leq r \leq \delta$ and $0 \leq \theta \leq 2\pi$, the sequence $\tilde{r}\,(\theta_0 + \theta + 2k\pi)$ is equicontinuous on $[0, 2\pi]$. Therefore, by Ascoli's Lemma, cf. Theorem 7.25 in Rudin [R], there exists a uniformly convergent subsequence of $\tilde{r}\,(\theta_0 + \theta + \; 2k\pi)$ converging to a solution $\tilde{r}_1(\theta)$ which satisfies $\tilde{r}_1(\theta) = \tilde{r}_1(\theta + 2k\pi)$; i.e., $\tilde{r}_1(\theta)$ is a non-zero periodic solution of (5). This contradicts the fact that

there are no closed trajectories of (3) in $N_\delta(\mathbf{0}) \sim \{0\}$ when the origin is not a center or a center focus of (3). Thus if the origin is not a center or a center focus of (3), $\tilde{r}_1 = 0$ and the origin is a focus of (3). This completes the proof of the theorem.

A center-focus cannot occur in an analytic system. This is a consequence of Dulac's Theorem discussed in Section 3.3 of Chapter 3. We therefore have the following corollary of Theorem 5 for analytic systems.

Corollary. Let E be an open subset of $\mathbf{R}^2$ containing the origin and let $\mathbf{f}$ be analytic in E with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Suppose that the origin is a center for the linear system (1) with $A = D\mathbf{f}(\mathbf{0})$. Then the origin is either a center or a focus for the nonlinear system (2).

As we noted in the previous section, Liapunov's method is one tool that can be used to distinguish a center from a focus for a nonlinear system. Another approach is to write the system in polar coordinates as in Examples 1-3 above. Yet another approach is to look for symmetries in the differential equations. The easiest symmetries to see are symmetries with respect to the x and y axes.

Definition 6. The system (3) is said to be symmetric with respect to the x-axis if it is invariant under the transformation $(t, y) \to (-t, -y)$; it is said to be symmetric with respect to the y-axis if it is invariant under the transformation $(t, x) \to (-t, -x)$.

Note that the system in Example 1 is symmetric with respect to the x-axis, but not with respect to the y-axis.

Theorem 6. Let E be an open subset of $\mathbf{R}^2$ containing the origin and let $\mathbf{f} \in C^1(E)$ with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. If the nonlinear system (2) is symmetric with respect to the x-axis or the y-axis, and if the origin is a center for the linear system (1) with $A = D\mathbf{f}(\mathbf{0})$, then the origin is a center for the nonlinear system (2).

The idea of the proof of this theorem is that by Theorem 5, any trajectory of (3) in $N_\delta(0)$ which crosses the positive x-axis will also cross the negative x-axis. If the system (3) is symmetric with respect to the x-axis, then the trajectories of (3) in $N_\delta(0)$ will be symmetric with respect to the x-axis and hence all trajectories of (3) in $N_\delta(0)$ will be closed; i.e., the origin will be a center for (3).

1.10 Nonhyperbolic Critical Points in R^2

In this section we present some results on nonhyperbolic critical points of planar analytic systems. This work originated with Poincaré [P] and was extended by Bendixson [B] and more recently by Andronov et al. [A – I]. We assume that the origin is an isolated critical point of the planar system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned} \tag{1}$$

where P and Q are analytic in some neighborhood of the origin. In Sections 2.9 and 2.10 we have already presented some results for the case when the matrix of the linear part $A = D\mathbf{f}(0)$ has pure imaginary eigenvalues, i.e., when the origin is a center for the linearized system. In this section we give some results established in [A − I] for the case when the matrix A has one or two zero eigenvalues, but $A \neq 0$. And these results are extended to higher dimensions in Section 2.12.

First of all, note that if P and Q begin with m th-degree terms P_m and Q_m, then it follows from Theorem 2 in Section 2.10 that if the function

$$g(\theta) = \cos\theta Q_m(\cos\theta, \sin\theta) - \sin\theta P_m(\cos\theta, \sin\theta)$$

is not identically zero, then there are at most $2(m+1)$ directions $\theta = \theta_0$ along which a trajectory of (1) may approach the origin. These directions are given by solutions of the equation $g(\theta) = 0$. Suppose that $g(\theta)$ is not identically zero, then the solution curves of (1) which approach the origin along these tangent lines divide a neighborhood of the origin into a finite number of open regions called sectors. These sectors will be of one of three types as described in the following definitions; cf. [A-I] or [L]. The trajectories which lie on the boundary of a hyperbolic sector are called separatrices. Cf. Definition 1 in Section 3.11.

Definition 1. A sector which is topologically equivalent to the sector shown in Figure 1(a) is called a hyperbolic sector. A sector which is topologically equivalent to the sector shown in Figure 1(b) is called a parabolic sector. And a sector which is topologically equivalent to the sector shown in Figure 1(c) is called an elliptic sector.

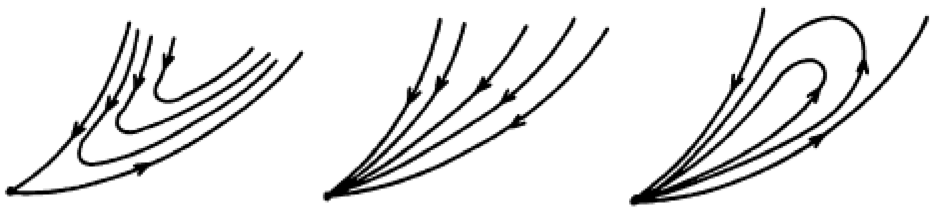

Figure 1. (a) A hyperbolic sector. (b) A parabolic sector. (c) An elliptic sector.

In Definition 1, the homeomorphism establishing the topological equivalence of a sector to one of the sectors in Figure 1 need not preserve the direction of the flow; i.e., each of the sectors in Figure 1 with the arrows reversed are sectors of the same type. For example, a saddle has a deleted neighborhood consisting of four hyperbolic sectors and four separatrices. And a proper node has a deleted neighborhood consisting of one parabolic sector. According to Theorem 2 below, the system

$$\dot{x} = y$$
$$\dot{y} = -x^3 + 4xy$$

has an elliptic sector at the origin; cf. Problem 1 below. The phase portrait for this system is shown in Figure 2. Every trajectory which approaches the origin does so tangent to the x-axis.

A deleted neighborhood of the origin consists of one elliptic sector, one hyperbolic sector, two parabolic sectors, and four separatrices. Cf. Definition 1 and Problem 5 in Section 3.11. This type of critical point is called *a* critical point with an elliptic domain; cf. [A-I].

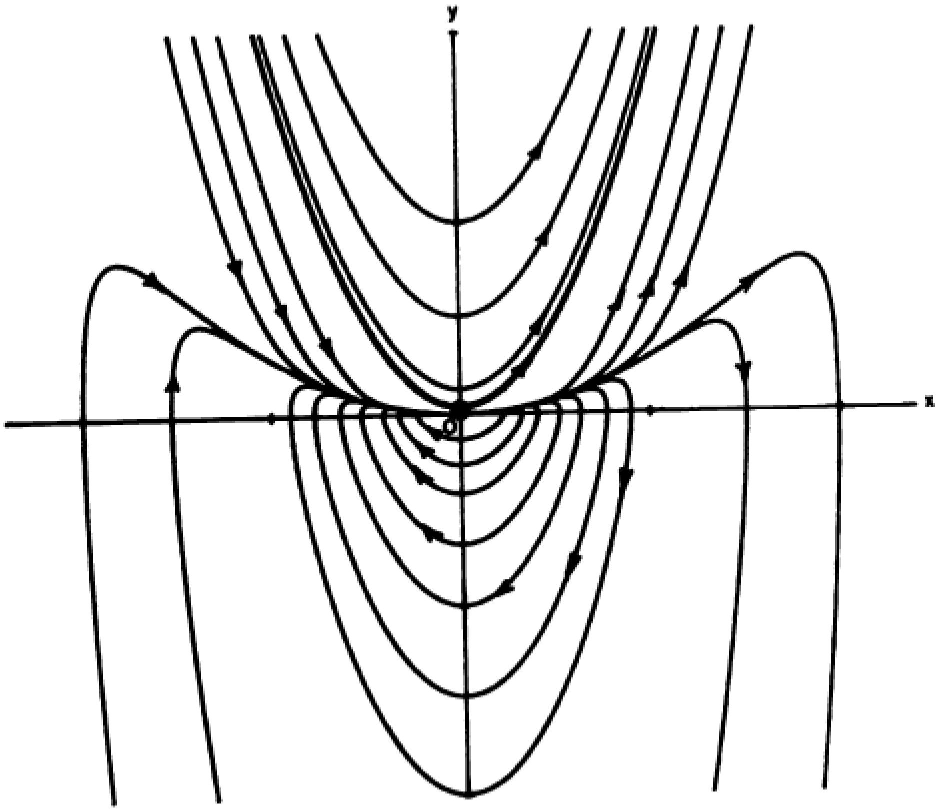

Figure 2. A critical point with an elliptic domain at the origin.

Another type of nonhyperbolic critical point for a planar system is a saddle-node. A saddle-node consists of two hyperbolic sectors and one parabolic sector (as well as three separatrices and the critical point itself). According to Theorem 1 below, the system

$$\dot{x} = x^2$$

$$\dot{y} = y$$

has a saddle-node at the origin. Even without Theorem 1, this system is easy to discuss since it can be solved explicitly for $x(t) = (1/x_0 - t)^{-1}$ and $y(t) = y_0 e^t$. The phase portrait for this system is shown in Figure 3.

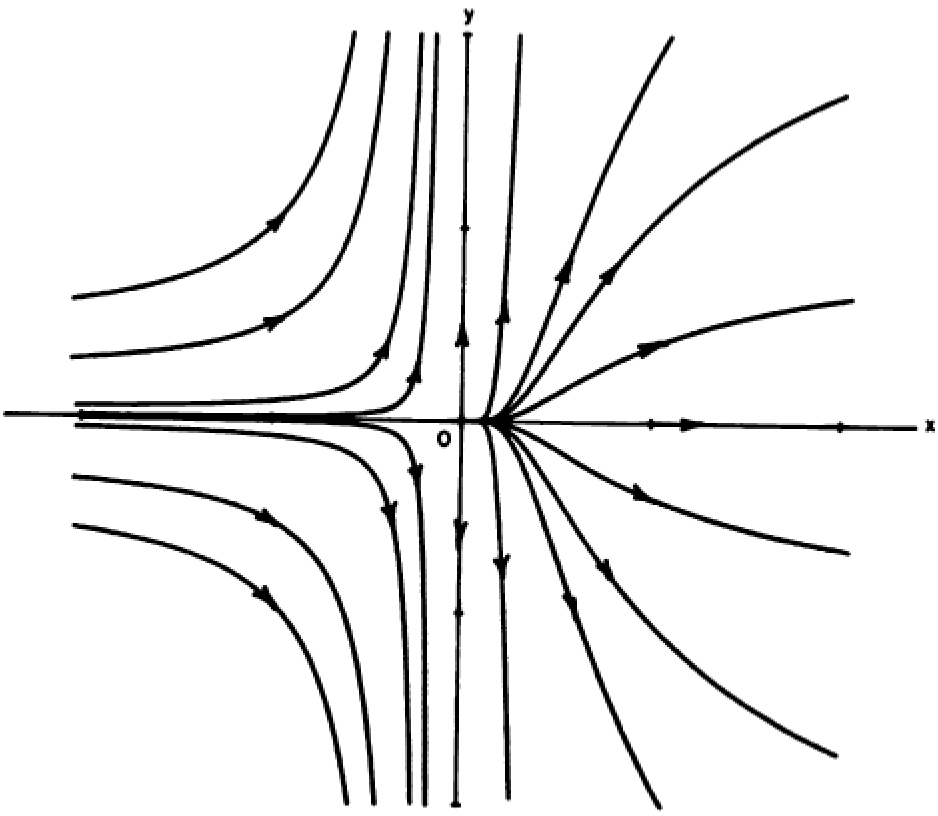

Figure 3. A saddle-node at the origin.

One other type of behavior that can occur at a nonhyperbolic critical point is illustrated by the following example:

$$\dot{x} = y$$
$$\dot{y} = x^2$$

The phase portrait for this system is shown in Figure 4. We see that a deleted neighborhood of the origin consists of two hyperbolic sectors and two separatrices. This type of critical point is called a cusp.

As we shall see, besides the familiar types of critical points for planar analytic systems discussed in Section 2.10, i.e., nodes, foci, (topological) saddles and centers, the only other types of critical points that can occur for (1) when $A \neq 0$ are saddle-nodes, critical points with elliptic domains and cusps.

We first consider the case when the matrix A has one zero eigenvalue, i.e., when $\det A = 0$, but $\operatorname{tr} A \neq 0$. In this case, as in Chapter 1 and as is shown in [A – I] on p. 338 , the system (1) can be put into the form

$$\dot{x} = p_2(x,y) \qquad (2)$$
$$\dot{y} = y + q_2(x,y)$$

where p_2 and q_2 are analytic in a neighborhood of the origin and have expansions that begin with second-degree terms in x and y. The following theorem is proved on p. 340 in [A-I]. Cf. Section 2.12.

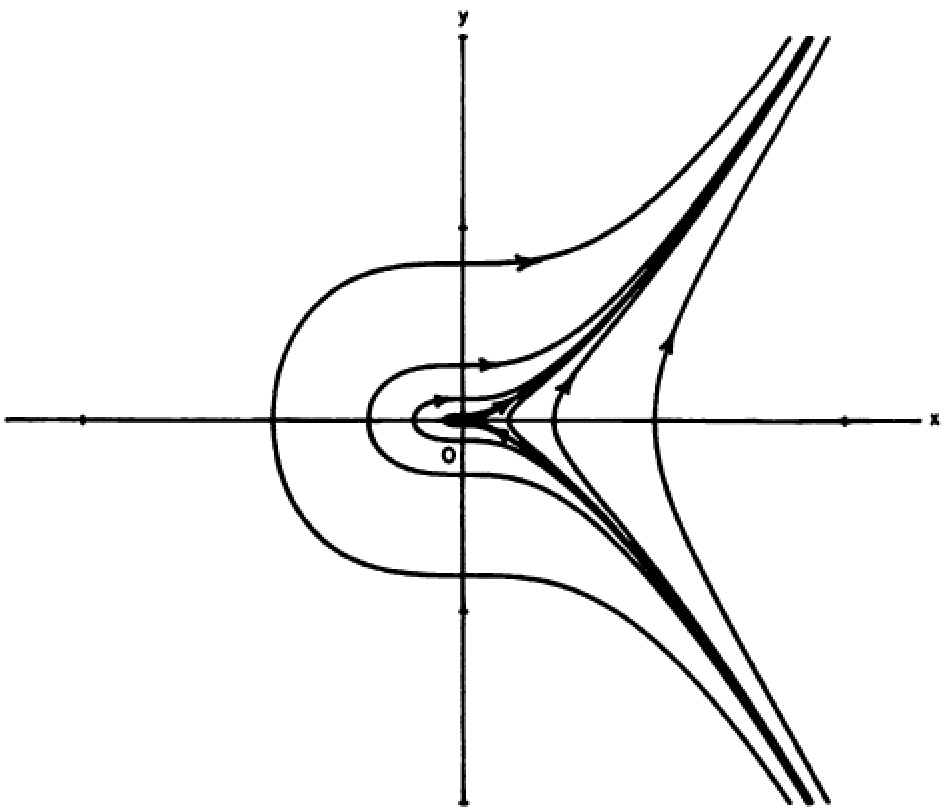

Figure 4. A cusp at the origin.

Theorem 1. Let the origin be an isolated critical point for the analytic system (2). Let $y = \phi(x)$ be the solution of the equation $y + q_2(x,y) = 0$ in a neighborhood of the origin and let the expansion of the function $\psi(x) = p_2(x, \phi(x))$ in a neighborhood of $x = 0$ have the form $\psi(x) = a_m x^m + \cdots$ where $m \geq 2$ and $a_m \neq 0$. Then (1) for m odd and $a_m > 0$, the origin is an unstable node, (2) for m odd and $a_m < 0$, the origin is a (topological) saddle and (3) for m even, the origin is a saddle-node.

Next consider the case when A has two zero eigenvalues, i.e., $\det A = 0$, $\operatorname{tr} A = 0$, but $A \neq 0$. In this case it is shown in [A-I], p. 356, that the system (1) can be put in the "normal" form

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= a_k x^k[1 + h(x)] + b_n x^n y[1 + g(x)] + y^2 R(x, y)\end{aligned} \tag{3}$$

where $h(x), g(x)$ and $R(x, y)$ are analytic in a neighborhood of the origin, $h(0) = g(0) = 0, k \geq 2, a_k \neq 0$ and $n \geq 1$. Cf. Section 2.13. The next two theorems are proved on pp. 357-362 in [A-I].

Theorem 2. Let $k = 2m + 1$ with $m \geq 1$ in (3) and let $\lambda = b_n^2 + 4(m + 1)a_k$. Then if $a_k > 0$, the origin is a (topological) saddle. If $a_k < 0$, the origin is (1) a focus or a center if $b_n = 0$ and also if $b_n \neq 0$ and $n > m$ or if $n = m$ and $\lambda < 0$, (2) a node if $b_n \neq 0, n$ is an even number and $n < m$ and also if $b_n \neq 0, n$ is an even number, $n = m$ and $\lambda \geq 0$ and (3) a critical point with an elliptic domain if $b_n \neq 0, n$ is an odd number and $n < m$ and also if $b_n \neq 0, n$ is an odd number, $n = m$ and $\lambda \geq 0$.

Theorem 3. Let $k = 2m$ with $m \geq 1$ in (3). Then the origin is (1) a cusp if $b_n = 0$ and also if $b_n \neq 0$ and $n \geq m$ and (2) a saddle-node if $b_n \neq 0$ and $n < m$.

We see that if $D\mathbf{f}(\mathbf{x}_0)$ has one zero eigenvalue, then the critical point $\mathbf{x}_0$ is either a node, a (topological) saddle, or a saddle-node; and if $D\mathbf{f}(\mathrm{x}_0)$ has two zero eigenvalues, then the critical point x_0 is either a focus, a center, a node, a (topological) saddle, a saddle-node, a cusp, or a critical point with an elliptic domain.

Finally, what if the matrix $A = 0$? In this case, the behavior near the origin can be very complex. If P and Q begin with m th-degree terms, then the separatrices may divide a neighborhood of the origin into $2(m + 1)$ sectors of various types. The number of elliptic sectors minus the number of hyperbolic sectors is always an even number and this number is related to the index of the critical point discussed in Section 3.12 of Chapter 3. For example, the homogenous quadratic system

$$\begin{aligned}\dot{x} &= x^2 + xy\\ \dot{y} &= \frac{1}{2}y^2 + xy\end{aligned}$$

has the phase portrait shown in Figure 5. There are two elliptic sectors and two parabolic sectors at the origin. All possible types of phase portraits for homogenous, quadratic systems have been classified by the Russian mathematician L.S. Lyagina [19]. For more information on the topic, cf. the book by Nemytskii and Stepanov [N/S].

Remark. A critical point, $\mathbf{x}_0$, of (1) for which $D\mathbf{f}(\mathbf{x}_0)$ has a zero eigenvalue is often referred to as a multiple critical point. The reason for this is made clear in Section 4.2 of Chapter 4 where it is shown that a multiple critical point of (1) can be made to split into a number of hyperbolic critical points under a suitable perturbation of (1).

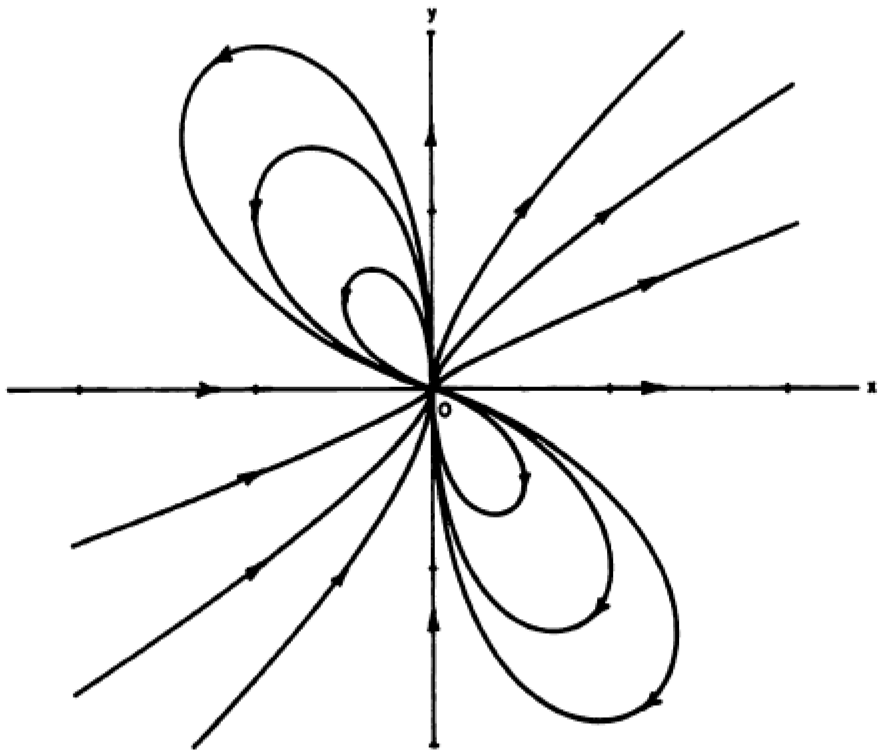

Figure 5. A nonhyperbolic critical point with two elliptic sectors and two parabolic sectors.

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