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Algebra I, Worksheet 3

Exercise n°1 : Which of the following relations define functions ?

$$\begin{aligned} a)\Gamma_{\mathfrak{R}_1} &= \{(2,1),(5,1),(8,1),(11,1),(14,1)\} \\ b)\Gamma_{\mathfrak{R}_2} &= \{(1,3),(1,5),(2,5)\} \\ c)\Gamma_{\mathfrak{R}_3} &= \{(1,c),(2,b),(3,a),(4,b)\} \end{aligned}$$

Exercise n°2 : Let $f : E \times E \longrightarrow \mathbb{R}$ be an application such that

$$\forall a,b,c \in E : f(a,b) + f(b,c) + f(c,a) = 0.$$

Prove that the relation $\mathfrak{R}$ defined on E by $a\mathfrak{R}b$ if and only if $f(a,b) = 0$ is an equivalence relation.

Exercise n°3 : Let E be a set, and A and B be two subsets of E. Prove the following properties

1. $\varphi_A + \varphi_{C_E^A} = 1$
2. $\varphi_{A\cap B} = \varphi_A.\varphi_B$
3. $\varphi_{A\cup B} = \varphi_A + \varphi_B - \varphi_A.\varphi_B$
4. $\varphi_{A\backslash B} = \varphi_A(1-\varphi_B)$.

where φ_A is the indicator mapping of A, defined as

$$\begin{aligned} \varphi_A : \ E &\longrightarrow \{0,1\} \\ x &\longmapsto \varphi_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases} \end{aligned}$$

Exercise n°4 :
1. Let $f : E \longrightarrow F$ be a mapping. Prove the following
(a) $\forall A,B \in \mathcal{P}(E) : A \subset B \Longrightarrow f(A) \subset f(B)$.
(b) $\forall A,B \in \mathcal{P}(E) : f(A\cup B) = f(A) \cup f(B)$.
(c) $\forall A,B \in \mathcal{P}(E) : f(A\cap B) \subset f(A) \cap f(B)$.
(d) $\forall C,D \in \mathcal{P}(F) : C \subset D \Longrightarrow f^{-1}(C) \subset f^{-1}(D)$.
(e) $\forall C,D \in \mathcal{P}(F) : f^{-1}(C\cup D) = f^{-1}(C) \cup f^{-1}(D)$.
2. Let the mapping $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = x - y$, and let $A = \{0,1\} \times \{1,2\}$.
a. Find $f(A)$. Deduce that f is not injective.
b. Is the mapping f surjective ?

Exercise n°5 : Let $E = [0,1]$ and $F = [0,2]$ be intervals in $\mathbb{R}$, and let f and g be two mappings defined as follows

$$\begin{aligned} f : \ E &\longrightarrow F \\ x &\longmapsto f(x) = 2 - x \end{aligned}, \qquad \begin{aligned} g : \ F &\longrightarrow E \\ x &\longmapsto g(x) = (x-1)^2 \end{aligned}$$

1. Determine the mappings $f \circ g$ and $g \circ f$.
2. Find $f^{-1}(\{0\})$. Deduce that f is not surjective.
3. Prove that $g \circ f$ is bijective, and find $(g \circ f)^{-1}$.

Exercise n°6 : Prove that the mapping

$$\begin{aligned} f : \ (\mathbb{N}^*, |) &\longrightarrow (\mathbb{N}^*, |) \\ x &\longmapsto f(x) = x^2 \end{aligned}$$

is strictly increasing with respect to the divisibility relation.

Exercise (Supplementary Exercise) Let E,F and G be three non-empty sets. Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be two mappings. Prove the following properties
1. If f and g are injective, then $g \circ f$ is injective.
2. If f and g are surjective, then $g \circ f$ is surjective.
3. If f and g are bijective, then $g \circ f$ is bijective.
4. If $g \circ f$ is injective, then f is injective.
5. If $g \circ f$ is surjective, then g is surjective.