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L3 Mathematics, S5.

Normed Vector Spaces
Exercises of chapter 1 :Hilbert spaces

Exercise1 Prove that in a real vector space with inner product we have :

$$\langle x/y\rangle = 1/4(\|x+y\|^2 - \|x-y\|^2)$$

and in a complex vector space with inner product we have

$$\langle x/y\rangle = 1/4(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

These are the so-called polarization identities. They tell us that in a Hilbert space, the inner product is determined by the norm.

Exercise 2 $E = \mathcal{M}_{(m,n)}(R)$ is the real vector space of matrices with m rows and n columns. For $a \in E, b \in E$, we put

$$\langle a/b\rangle = tr(a^T.b).$$

- Show that we define an inner product on E.

Exercise 3 Show that the sup-norm on $\mathcal{C}[a,b]$ the vector space of all $\mathbb{C}-$valued continuous functions on $[a,b]$, is not induced by an inner product .

we have:

$$\|f\| = max_{t\in[a,b]}|f(t)|.$$

Exercise 4* Let H be an inner product space. Describe all pairs of vectors x, y for which

$$\|x+y\| = \|x\| + \|y\|$$

Exercise 5* Let $[a,b]$ be a finite interval. Show that $L^2([a,b]) \subset L^1([a,b])$.

Exercise 6

1. Let H be a Hilbert space, and let B be the closed unit ball of H.
   (a) Show that $\forall x \in H \setminus B, \forall z \in B$, we have $\left(\mathrm{Re}\left\langle z, \frac{x}{\|x\|}\right\rangle - 1\right) \leq 0$.
   (b) Deduce the sign of $\mathrm{Re}\left\langle z - \frac{x}{\|x\|}, x - \frac{x}{\|x\|}\right\rangle$.
   (c) Derive an expression for the projection onto B, the closed unit ball of H. Justify.
2. We consider in the space of periodic functions $L_2[-\pi,\pi]$, the subspace $F = vect\left\{e^{-int}, ..., e^{int}\right\}$.
   (a) Find the projection of f on F
   (b) Deduce the distance from f to the subspace F.

Exercise 7 Let be $H = \ell^2(\mathbb{N},\mathbb{R})$ (the real Hilbert space). We denote

$$C = \{x = (x_n) \in H; \forall n \in \mathbb{N},\ x_n \geq 0\}.$$

1 – Prove that C is a closed convex set.
2 – Determine the projection on this convex C.
3 – Resume the previous question with $H = \ell^2(\mathbb{N},\mathbb{C})$

Exercise 8 Let E be the inner product space of complex sequences $(u_n)_{n\in\mathbb{N}}$ satisfying :

$\exists\ N \in \mathbb{N}, \forall n \geq N, \quad u_n = 0$

with the inner product $\langle u/v \rangle = \sum_{n=0}^{+\infty} u_n \overline{v_n}$.

1 - Show that the mapping $\varphi(u) : E \mapsto C$ defined by $\varphi(u) = \sum_{n=1}^{+\infty} \frac{u_n}{n}$ is a linear continuous map on E.

2 - Is there an element $a \in E$ such that for all u in E, we have $\varphi(u) = \langle u/a \rangle$?

3 - What can we deduce about E ?

Exercise 9 $(H, \langle ., . \rangle)$ is a Hilbert space. F a closed subspace. Demonstrate that:

1- $\|P_F\| = 1$

2- $Ker\ P_F = F^{\perp}$

3- $F = \{x \in H, P_F(x) = x\}$

Exercise 10 For all $N \in \mathbb{N}$, note by M_N the vector subspace of $\ell^2(\mathbb{N}, \mathbb{C})$ formed with sequences $(x_n)_{(n\in N)}$ such that $\sum_{n=0}^{N} x_n = 0$.

1 - Show that the mapping $(x_n)_n \mapsto \sum_{k=0}^{N} x_k$ is linear continuous from $\ell^2(\mathbb{N}, \mathbb{C})$ to $\mathbb{C}$. What can we deduce about M_N ? Conclude that $\ell^2(\mathbb{N}, \mathbb{C}) = M_N \oplus M_N^{\perp}$.

2 - Let be $E = \{(y_n)_n$ such that, for all $0 \leq i \leq j \leq N$, we have $y_i = y_j$ and $y_n = 0$ for $n > N\}$

3 - Show that the orthogonal $M_N^{\perp}$ of M_N contains E.

4 - Show that $M_N^{\perp} = E$ (note that, for $0 \leq i \leq j \leq N$, the sequence (x_n)such that $x_i = 1, x_j = -1$ and $x_n = 0$ if $n \neq i$ and $n \neq j$ belongs to M_N

Exercise 11 Find the Fourier coefficients of the following functions:

(a)$f(t) = t$

(b) $^* f(t) = t^2$

(c) $^* \cos at\ t \in \mathbb{R}\backslash\mathbb{Z}$ ($\mathbb{Z}$ is the set of integers)

(d) $f(t) = |t|$

Use the Parseval equality to prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Find $\sum_{n=1}^{\infty} \frac{1}{n^4}$

Exercise12 Let $f(x)$ be a ditferentiable $2\pi-$periodic function in $[-\pi, \pi]$ with derivative $f'(x) \in L_2[-\pi, \pi]$. Let f_n for $n \in \mathbb{Z}$ be the Fourier coeffi-

cients of $f(x)$ in the system $\left\{e^{inx}/\sqrt{2\pi}\right\}$. Prove that $\sum_{n\in\mathbb{Z}} |f_n| < \infty$.