CHAPTER 2: IMPROPER INTEGRALS

Improper Integral of the Second Kind:

If $f(x)$ is unbounded only at the endpoint $x = a$ of the interval $a \leq x \leq b$, then we define:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$$

We say that $\int_a^b f(x)\,dx$ is convergent if the limit exists.

We say that $\int_a^b f(x)\,dx$ is divergent if the limit does not exist.

If $f(x)$ is unbounded only at the endpoint $x = b$ of the interval $a \leq x \leq b$, then we define:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)\,dx$$

We say that $\int_a^b f(x)\,dx$ is convergent if the limit exists.

We say that $\int_a^b f(x)\,dx$ is divergent if the limit does not exist.

If $f(x)$ is unbounded only at an interior point $x = x_0$ of the interval $a \leq x \leq b$. Then, we set:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon_1 \to 0^+} \int_a^{x_0-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2 \to 0^+} \int_{x_0+\varepsilon_2}^b f(x)\,dx$$

We say that $\int_a^b f(x)\,dx$ is convergent if the limit exists.

We say that $\int_a^b f(x)\,dx$ is divergent if the limit does not exist.

Cauchy Principal Value:

It may happen that the limits do not exist when ε_1 and ε_2 tend to zero independently. In this case, it is possible to set $\varepsilon_1 = \varepsilon_2 = \varepsilon$ so that:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_a^{x_0-\varepsilon} f(x)\,dx + \lim_{\varepsilon_2 \to 0^+} \int_{x_0+\varepsilon}^b f(x)\,dx$$

If this limit exists, we call this limiting value the **Cauchy Principal Value**.

Improper Integral of the Second Kind for Specific Functions:

- $\int_a^b \frac{dx}{(x-a)^p}\,dx$ converges if $p < 1$, and diverges if $p \geq 1$.
- $\int_a^b \frac{dx}{(b-x)^p}\,dx$ converges if $p < 1$, and diverges if $p \geq 1$.

Remark: In the case where $p \leq 0$, the integrals are proper.

Convergence Criteria for Improper Integrals of the Second Kind:

- Comparison Criterion for Integrals with Non-negative Integrand:
  a) Convergence: Let $g(x) \geq 0$ for all $a < x \leq b$, and suppose that $\int_a^b g(x)\,dx$ converges. Then, if $0 \leq f(x) \leq g(x)$ for all $a < x \leq b$, it follows that $\int_a^b f(x)\,dx$ also converges.
  b) Divergence: Let $g(x) \geq 0$ for all $a < x \leq b$, and suppose that $\int_a^b f(x)\,dx$ diverges. Then, if $f(x) \geq g(x)$ for all $a < x \leq b$, it follows that $\int_a^b f(x)\,dx$ also diverges.
- Quotient Criterion for Integrals with Non-Negative Integrand:
  a) If $f(x) \geq 0$ and $g(x) \geq 0$, for all $a < x \leq b$, if $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = A \neq 0$ or ∞, then $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ both converge or both diverge.
  b) If $A = 0$ dans (a) and if $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converges.
  c) If $A = \infty$ dans (a) and if $\int_a^{+\infty} g(x)\,dx$ diverges, then $\int_a^{+\infty} f(x)\,dx$ diverges.

This criterion is related to the comparison criterion, of which it is a very useful alternative form. In particular, by taking $g(x) = \frac{1}{(x-a)^p}$, we obtain, based on the known behavior of this integral:

Theorem 2 : let $\lim_{x \to +\infty} (x-a)^p f(x) = A$, then :

- $\int_a^b f(x)\,dx$ converges if $p < 1$ and if A is finite.
- $\int_a^b f(x)\,dx$ diverges if $p \geq 1$ and if $A \neq 0$ (A is infinite).

Theorem 3 : let $\lim_{x \to +\infty} (b-x)^p f(x) = B$, then:

- $\int_a^b f(x)\,dx$ converges if $p < 1$ and if B is finite.
- $\int_a^b f(x)\,dx$ diverges if $p \geq 1$ and if $B \neq 0$ (A is infinite).

- Absolute and Conditional Convergence: $\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b |f(x)|\,dx$ converges. If $\int_a^b f(x)\,dx$ converges but $\int\int_a^b |f(x)|\,dx$ diverges, then $\int\int_a^b f(x)\,dx$ is said to be conditionally convergent.