Chapter 1: Simple Integrals and Multiple Integrals

Triple Integral

Let $F\ (x, y, z)$ be a function defined in a closed and bounded domain D in the $Ox, Oy, Oz.$ space. We arbitrarily divide the domain D into n elementary domains with volumes $\Delta V_1, \Delta V_2, \ldots, \Delta V_n$. Now, we choose an arbitrary point$(Pk)_{1\le k\le n}$ in each elementary domain. Let $F(P_1), F(P_2), \ldots, F(P_n)$ be the values of the function $F\ (x, y, z)$ at these points, and then form the products $F(P_k)\Delta V_k$.

The integral sum of the function $F(x, y, z)$ over the domain D is defined as a sum of the form

$$\sum_{k=1}^{n} F(P_k)\Delta V_k = F(P_1)\Delta V_1 + F(P_2)\Delta V_2 + \cdots + F(P_n)\Delta V_n$$

The triple integral of the function $F(x, y, z)$ over the domain D is defined as the limit of the integral sum when the largest of the volumes $\Delta V_k \to\ 0$. It is denoted by

$$\iiint F\ (x, y, z)dxdydz = \iiint F(P_k)dV = \lim_{max\Delta V_k \to 0} \sum_{k=1}^{n} F(P_k)\Delta V_k$$

Rule for Calculating a Triple Integral

Let F be a function defined and continuous in a closed domain D of $\mathbb{R}^3$, then:

$$\iiint F(x, y, z)\, dxdydz = \iint dxdy \int_{z_1}^{z_2} F(x, y, z)dz$$

The double integral over $dxdy$ is calculated on the domain T, where T is the orthogonal projection of D on to the xOy plane.

Example: calculate the triple integral $I_1 = \iiint (x + y + z)\, dxdydz$ the domain D is defined by:

$$D = \{(x, y, z) \in \mathbb{R}^3, x > 0, y > 0, z > 0, x + y + z \le 2\}$$

the orthogonal projection of the domain D on to the xOy plane, where $z\ =\ 0$, is

$$T = \{(x, y) \in \mathbb{R}^2, x > 0, y > 0, x + y \le 2\}$$

Additionally, $z_1 = 0\ et\ z_2 = 2 - x - y$ et $z_1 \leq z \leq z_2$

So,

$$I_1 = \iint dxdy \int_0^{2-x-y} (x + y + z)dz = \iint \left[(x + y)z + \frac{1}{2}z^2\right]_0^{2-x-y}$$

$$= \iint [2x + 2y - x^2 - y^2 - 2xy + \frac{1}{2}(2 - x - y)^2]dxdy$$

We have :

$$\begin{cases} 0 < y < 2 - x \\ 0 < x \leq 2 \end{cases}$$

Thus, the double integral of this function is:

$$I_1 = \int_0^2 \left[\int_0^{2-x} 2x + 2y - x^2 - y^2 - 2xy + \frac{1}{2}(2 - x - y)^2 dy\right] dx$$

$$= \int_0^2 [(2 - x)xy + y^2 - \frac{1}{6}y^3 - xy^2 + \frac{1}{2}(2 - x^2)y - \frac{1}{2}(2 - x)y^2]_0^{2-x}\, dx$$

$$= \int_0^2 [(2 - x)^2 - \frac{(2 - x)^3}{6}]\, dx$$

$$= [-\frac{(2 - x)^3}{3} + \frac{(2 - x)^4}{24}]_0^2 = \frac{1}{3} - \frac{1}{24} = \frac{7}{24}$$

Change of Variables in a Triple Integral

The change of variables in a triple integral allows for transitioning from the variables *x*, *y*, *z* to new variables *u*, *v*, *w* related to the original variables by the following relations:

$$x = x(u, v, w),$$

$$y = y(u, v, w),$$

$$z = z(u, v, w),$$

Where $x = x(u, v, w), y = y(u, v, w)\ and\ z = z(u, v, w)$, and their first partial derivatives are continuous functions in a domain D' . The Jacobian of the transformation in the domain D' is:

$$J = \left|\frac{D(x, y, z)}{D'(u, v, w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$$

In this case, the formula for transforming a triple integral is

$$\iiint F(x, y, z)\, dxdydz = \iiint F(x(u, v, w), y(u, v, w), z(u, v, w)|J|dudvdw$$

Triple Integral in Spherical Coordinates

Let $M(x, y, z)$ be a point in space $\mathbb{R}^3$. We then have the following transformation formula:

$$x = r \cos\theta \sin\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\varphi$$

with $> 0, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi$.

the jacobien is:

$$J = \begin{vmatrix} \cos\theta \sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\theta \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\sin\varphi \end{vmatrix} = r^2 \sin\varphi$$

It follows that: $dxdydz = r^2 \sin\varphi \, drd\theta d\varphi$

Example: Calculate $I_2 = \iiint x^2 \, dxdydz$, Where the domain D is defined by:

$$D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 \leq a^2, a > 0\}$$

We perform a change of variables to spherical coordinates.

Now, let (x, y, z) be a point in D, we have:

$$x^2 + y^2 + z^2 \leq a^2 \text{ , } r^2 \leq a^2 \text{ , and } 0 \leq r \leq a$$

then:

$$I_2 = \iiint x^2 \, dxdydz = \int_0^a r^4 \, dr \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^{\pi} \sin^3\varphi \, d\varphi$$
$$= \left[\frac{1}{5}r^5\right]_0^a \int_0^{2\pi} \left(\frac{1 + \cos 2\theta}{2}\right) \int_0^{\pi} \sin^3\varphi \, d\varphi$$
$$= \frac{4}{15}\pi a^5$$

Triple Integral in Cylindrical Coordinates

The transition from Cartesian coordinates to cylindrical coordinates is related by the following relations:

$$x = r \, cos \, \theta$$
$$y = r \, sin \, \theta$$
$$z = z$$

with $r > 0 \; et \; 0 \leq \theta \leq 2\pi$. the value of jacobien is : $J = r$

$$\iiint F(x, y, z) \, dxdydz = \iiint F(r\cos\theta, r\sin\theta, z) \, rdrd\theta dz$$

Volume Calculation

The volume of a body occupying the domain D is given by the following formula:

$$V = \iiint dxdydz$$