Chapter 1: Simple Integrals and Multiple Integrals

Multiple Integrals and Area Calculation

Double Integral:

Let $z = F(x,y)$ be a function defined in a closed and bounded domain D of the xoy plane. We decompose the domain D arbitrarily into n elementary domains of surfaces $\Delta_{s_1}, \Delta_{s_2}, \dots, \Delta_{s_n}$. Now, we choose an arbitrary point $(P_k)_{1\leq k\leq n}$in each elementary domain.

Let $F(P_1), F(P_2), \dots, F(P_n)$ be the values of the function $F(x,y)$ at these points, and then form the products $F(P_k)\Delta_{s_k}$. The sum of these products is called the integral sum of the function $F(x,y)$ over the domain D, and it takes the following form:

$$\sum_{k=1}^{n} F(P_k)\Delta_{s_k} = F(P_1)\Delta_{s_1} + F(P_2)\Delta_{s_2} + \cdots + F(P_n)\Delta_{s_n}$$

The double integral of the function $F(x,y)$ over the domain D is defined as the limit of the integral sum when the largest of the domains $\Delta_{s_k} \to 0$. It is denoted as:

$$\iint F(x,y)\, dxdy = \iint F(P)\, ds = \lim_{max\Delta_{s_k}\to 0} \sum_{k=1}^{n} F(P_k)\, \Delta_{s_k}$$

Rule for calculating a double integral:

Let $F(x,y)$ be a function defined and continuous in a closed domain D of $\mathbb{R}^2$, and this domain is such that:

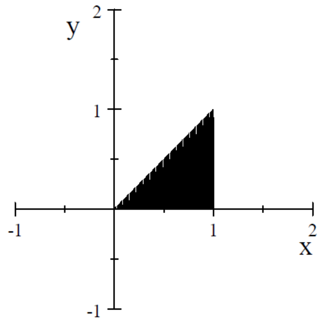
$$\begin{cases} \text{si } a \leq x \leq b & \Longrightarrow \quad f_1(x) \leq y \leq f_2(x) \\ & ou \\ \text{si } c \leq y \leq d & \Longrightarrow \quad g_1(y) \leq x \leq g_2(y) \end{cases}$$

Then, the double integral of the function F over D is calculated as follows:

$$\iint F(x,y)\, dxdy = \int_a^b [\int_{f_1(x)}^{f_2(x)} F(x,y)\, dy]dx = \int_c^d [\int_{g_1(y)}^{g_2(y)} F(x,y)dx]dy$$

Example: Calculate the following double integral: $I_1 = \iint \frac{x}{\sqrt{y}} dxdy$

Where the domain D is defined by: $D = \{(x,y) \in \mathbb{R}^2 / 0 < y \leq x < 1\}$



Let $(x, y) \in D$, we choose $0 < x < 1$, and it follows that $0 < y \leq x$. Thus,

$$I_1 = \int_0^1 \left[\int_0^x \frac{x}{\sqrt{y}} dy\right] dx = \int_0^1 x[2\sqrt{y}]_0^x \, dx = 2\int_0^1 (x\sqrt{x}) \, dx = 2(\frac{2}{5}x^{5/2})_0^1 = \frac{4}{5}$$

Fubini's Theorem or Integration over a Rectangle:

Let F(x, y) be a function defined and continuous in a rectangle $D = [a, b] \times [c, d]$ of $\mathbb{R}^2$:

Then,

$$\iint F(x, y) \, dxdy = \int_a^b [\int_c^d F(x, y) \, dy] dx = \int_c^d [\int_a^b F(x, y) dx] dy$$

Example: Calculate the following double integral: $I_2 = \iint (x^2 + y) \, dxdy$

If the domain $D = [0,1] \times [1,2]$:

According to Fubini's Theorem, we have:

$$\iint (x^2 + y) \, dxdy = \int_0^1 \left[\int_1^2 (x^2 + y) \, dy\right] dx = \int_0^1 [x^2 y + \frac{1}{2}y^2]_1^2 \, dx = \int_0^1 (x^2 + \frac{3}{2}) \, dx$$

$$= \left(\frac{1}{3}x^3 + \frac{3}{2}x\right)_0^1 = \frac{11}{6}$$

Change of Variables in a Double Integral

Double Integral in Curvilinear Coordinates:

Suppose the integration variables x and y can be expressed as functions of new variables u and v as follows:

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$$

Where the functions $x(u, v)$ and $y(u, v)$ have continuous partial derivatives in a domain D' of the $uo'v$ plane, and the Jacobian of the transformation in the domain D' does not cancel:

$$J = \left|\frac{D(x,y)}{D(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

Then, the formula for transforming a double integral to curvilinear coordinates is as follows:

$$\iint F(x,y)\,dxdy = \iint F(x(u,v),y(u,v))|J|\,dudv$$

Example: Perform the change of variable indicated in the following integral:

$$I_4 = \int_0^1 dx \int_x^{2x} F(x,y)\,dy, \qquad \begin{cases} u = x+y \\ v = \frac{y}{x+y} \end{cases}$$

The new values of x and y in terms of (u,v) are:

$$\begin{cases} x = u - uv \\ y = uv \end{cases}$$

So,

$$J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

The new domain D' is:

$$D' = \{(u,v) \in \mathbb{R}^2, \frac{1}{2} \leq v \leq \frac{2}{3}, \qquad 0 \leq u \leq \frac{1}{1-v}\}$$

Then the integral I_4 is:

$$I_4 = \int_{\frac{1}{2}}^{\frac{2}{3}} dv \int_0^{\frac{1}{1-v}} uG(u,v)\,du$$

Double Integral in Polar Coordinates

When transitioning from rectangular (Cartesian) coordinates x and yyy to polar coordinates r and θ, we define:

$$\begin{cases} x = rcos\theta \\ y = rsin\theta \end{cases} \text{avec } r > 0 \; et \; 0 \leq \theta \leq 2\pi$$

On a : $J = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r$

The double integral of a function F in polar coordinates is:

$$\iint F(x,y)\,dxdy = \iint F(rcos\theta, rsin\theta)\,rdrd\theta$$

Example : Calculate :$I_5 = \iint \frac{dxdy}{x^2+y^2}$ where the domain D is defined by :

$$D == \{(x,y) \in \mathbb{R}^2, 4 \leq x^2 + y^2 \leq 9\}$$

We have the polar coordinates as: $\begin{cases} x = rcos\theta \\ y = rsin\theta \end{cases}$

$$4 \leq x^2 + y^2 \leq 9 \, , 2 \leq r \leq 3 \; de \; plus \; 0 \leq \theta \leq 2\pi$$

We have :

$$I_5 = \iint \frac{dxdy}{x^2 + y^2} = \iint \frac{rdrd\theta}{r^2cos^2\theta + r^2sin^2\theta} = \int_2^3 \frac{1}{r} dr \int_0^{2\pi} d\theta = 2\pi[\ln r]_2^3 = \pi \ln \frac{9}{4}$$

Area Calculation in $\mathbb{R}^2$

The area of a plane figure bounded by the domain D is calculated using the formula $A(D)$:

$$A(D) = \iint dxdy$$

Area Calculation in $\mathbb{R}^3$

Let S be a uniform surface given by the equation $z = f(x; y)$; then the area of the surface S is expressed by the formula:

$$A(S) = \iint \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{dy}\right)^2} \, dxdy$$