Chapter 1: Simple and Multiple Integrals

1. Review of the Riemann Integral and Calculation of Primitives

Subdivision: Let $[a,b]$ be a finite interval. We call a subdivision S of $[a,b]$ any finite ordered sequence $(t_i)_{0\leq i\leq n}$ within $[a,b]$.

$$a = t_0 \leq t_1 \leq \cdots \leq t_n = b$$

The '**step**' of a subdivision S is defined as:

$$\rho(S) = max_{0\leq i\leq n-1}|t_{i+1} - t_i|$$

The uniform subdivision of the interval [a,b] is $\frac{b-a}{n}$, and each point is calculated by the following equation:

$$t_i = a + i\frac{b-a}{n}, 0 \leq i \leq n$$

Riemann Integral

By definition, the Riemann integral of the function f is:

$$\int_a^b f(x)\,dx = Sup_S A^+(f,S) = Inf_S A^-(f,S)$$

A function $f: [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if, for every $\varepsilon > 0$, there exists a subdivision S such that its Darboux sums satisfy

$$A^+(f,S) - A^-(f,S) \leq \varepsilon$$

The lower Darboux sum:

$$A^-(f,S) = \sum_{i=0}^{n-1} m_i(t_{i+1} - t_i)$$

With:

$$m_i = inf_{x\in[t_i,t_{i+1}[}\, f(x)$$

The upper Darboux sum:

$$A^+(f,S) = \sum_{i=0}^{n-1} M_i(t_{i+1} - t_i)$$

With:

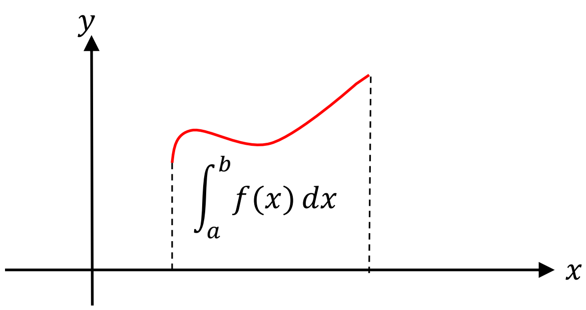
$$M_i = sup_{x\in[t_i,t_{i+1}[}\, f(x)$$

If the subdivision is uniform, the Riemann sum takes the following form

$$R(f,S_{unif}) = \lim_{n\to+\infty} \frac{b-a}{n}\sum_{i=0}^{n-1} f(a + i\frac{b-a}{n}) = \frac{b-a}{n}\sum_{i=0}^{n-1} f(a + i\frac{b-a}{n})$$

Calculation of Primitives

Let f be a continuous and positive function on the interval $[a,b]$, the integral of this function is the area expressed in square units of the surface bounded by the curve C_f and the x-axis, between the points $x = a$ and $x = b$.



We consider a function f continuous on an interval I. We say that a function F is a primitive of f on I if F is differentiable and its derivative is equal to:

$$F(x)' = f(x)$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Primitives of common functions:

We have c a real constant.

- $\int 1\, dx = x + c$
- $\int x^m\, dx = \frac{x^{m+1}}{m+1} + c$
- $\int \cos(x)\, dx = sin(x) + c$
- $\int \frac{1}{\sqrt{a^2-x^2}} dx = arcsin(\frac{x}{a}) + c$
- $\int \text{tang}(x)\, dx = -\ln(\cos(x)) + c$
- $\int \frac{1}{x} dx = \ln|x| + c$
- $\int e^x\, dx = e^x + c$
- $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} arctang(\frac{x}{a}) + c$
- $\int \frac{1}{sin^2(x)} dx = cotg(x) + c$
- $\int \frac{1}{cos^2(x)} dx = tang(x) + c$

The following functions all follow the rules of differentiation:

$$\int [u(x)]^n u'(x)\, dx = \frac{[u(x)]^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int e^{u(x)} u'(x)\, dx = e^{u(x)} + c$$

$$\int \frac{u'(x)}{u(x)} dx = \ln|u(x)| + c$$

$$\int \sin\big(u(x)\big)\, u'(x)\, dx = -\cos\big(u(x)\big) + c$$

$$\int \cos\big(u(x)\big)\, u'(x)\, dx = \sin\big(u(x)\big) + c$$

$$\int \frac{u'(x)}{\sqrt{u(x)}} dx = -\frac{1}{u(x)} + c$$

$$\int a^x\, dx = \frac{a^x}{\ln(a)} + c$$

$$\int u'(x) tang\big(u(x)\big)\, dx = -\ln(\cos(x)) + c$$

Some useful trigonometric formulas:

- $tang(x) = \frac{\sin(x)}{\cos(x)}$
- $cotg(x) = \frac{\cos(x)}{\sin(x)}$
- $cos^2(x) + sin^2(x) = 1$
- $cos^2(x) = \frac{1+\cos(2x)}{2}$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $sin^2(x) = \frac{1-\cos(2x)}{2}$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
- $\cos(2x) = 2cos^2(x) - 1 = 1 - 2sin^2(x)$

Integration by parts:

Let u and v be two continuous functions on $[a,b]$, then:

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,dx$$

Example: Calculate the following integral:

$$\int_0^{\frac{\pi}{2}} xsin(x)\,dx$$

We set: $u(x) = x$, $u'(x) = 1$ $et\ v'(x) = \sin(x)$, $v(x) = -\cos(x)$

$$\int_0^{\frac{\pi}{2}} xsin(x)\,dx = [-xcos(x)]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} -\cos(x)\,dx = [-xcos(x)]_0^{\pi/2} + [\sin(x)]_0^{\pi/2} = 1$$

Variable change:

Let $f: [a,b] \to \mathbb{R}$ be a continuous function. Suppose there exists a function $v: [a,b] \to \mathbb{R}$ of class C^1 and a function u, continuous on $v([a,b])$, such that for all x in $[a,b]$, the following equality holds:

$$f(x) = u(v(x))v'(x)$$

Then:

$$\int_{a}^{b} f(x)\, dx = \int_{v(a)}^{v(b)} u(y)\, dy$$

In practical calculations, we set: $y = v(x), d'où\ dy = v'(x)dx$

Finally, we change the limits of the integral.

Example: calculate the following integral using a change of variable:

$$\int_{0}^{2} \frac{2e^{\sqrt{x}}}{\sqrt{x}} dx$$

We use the following change of variable:

$$y = v(x) = \sqrt{x}$$

We obtain:

$$v(a) = 0, v(b) = \sqrt{2}\ et\ v'(x) = \frac{1}{2\sqrt{x}}$$

$$\int_{0}^{2} \frac{2e^{\sqrt{x}}}{\sqrt{x}} dx = 4\int_{0}^{2} \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx = 4\int_{0}^{\sqrt{x}} e^{y}\, dy = [e^{y}]_{0}^{\sqrt{2}} = 4(e^{\sqrt{2}} - 1)$$

Primitives of rational fractions:

A rational fraction is defined as the quotient of two polynomials. Most of the primitives that we can formally calculate can be done using simple changes of variables.

- Integral of the type: $\int \frac{1}{x+\alpha} dx, \alpha \in \mathbb{R}$

$$\int \frac{1}{x+\alpha} dx = \ln|x+\alpha| + c, c \in \mathbb{R}$$

- Integral of the type: $\int \frac{1}{(x+\alpha)^{n}} dx, \alpha \in \mathbb{R}\ and\ n \geq 1$

$$\int \frac{1}{(x+\alpha)^n}dx = \frac{1}{(1-n)(x+\alpha)^{n-1}} + c, c \in \mathbb{R}$$