Introduction:

We will consider the case of fluids at rest or in motion without relative movement between adjacent particles. In both cases there are no friction forces (no shear stress between the fluid particles) in the fluid. All forces that develop on surfaces are due to pressure.

The absence of friction simplifies the analysis and makes it possible to obtain relatively simple solutions to some problems of practical interest.

Therefore the types of force are:

- Surface forces due to pressure
- Volume forces due to weight.

II.1 Basic law of hydrostatics

Let us consider a volume element of parallelepedic shape inside a fluid in equilibrium, of volume dV=dx.dy.dz in a Cartesian coordinate system and take stock of the forces which apply to this volume element $\sum \overrightarrow{F_{ext}} = \vec{0}$:

- The volume force: the weight of the fluid given by :

$$\overrightarrow{\boldsymbol{dP}} = \boldsymbol{dm}.\vec{\boldsymbol{g}} = \rho dV\, \vec{g} = \rho\, dx.dy.dz.\vec{g}$$

***Note:** be careful, ρ dV g designates volume forces. It is often gravity, but there can be other forces, particularly in relative reference where the training inertia and Coriolis forces must be taken into account.*

- The surface forces due to pressure : $\overrightarrow{\boldsymbol{dF}} = \vec{\boldsymbol{P}}\boldsymbol{dS}$ we can decompose the resultant into three components dF_x, dF_y, dF_z in the three directions (x,y,z).

$$\overrightarrow{dF} = dF_x.\vec{\imath} + dF_y.\vec{\jmath} + dF_z.\vec{k}$$

- Since the surface forces are necessarily normal, the component following **x** corresponds to the pressure forces exerted on the surfaces perpendicular to the axis **x**.

The resultant of the pressure forces along the axis **(ox)** is :

$$dF_x = P(x,y,z)dydz - p(x+dx,y,z)dydz$$

By a first order development, we have:

$P(x+dx) = p(x) + \frac{\partial P}{\partial x}dx + \cdots$ D'où : $dF_x = -\frac{\partial P}{\partial x}dxdydz.$

The resultant of the pressure forces along the axis **(oy)** is:

$$dF_y = P(x,y,z)dxdz - p(x,y+dy,z)dxdz$$

By a first order development, we have:

$P(y+dy) = p(y) + \frac{\partial P}{\partial y}dy + \cdots$ D'où : $dF_y = -\frac{\partial P}{\partial y}dxdydz.$

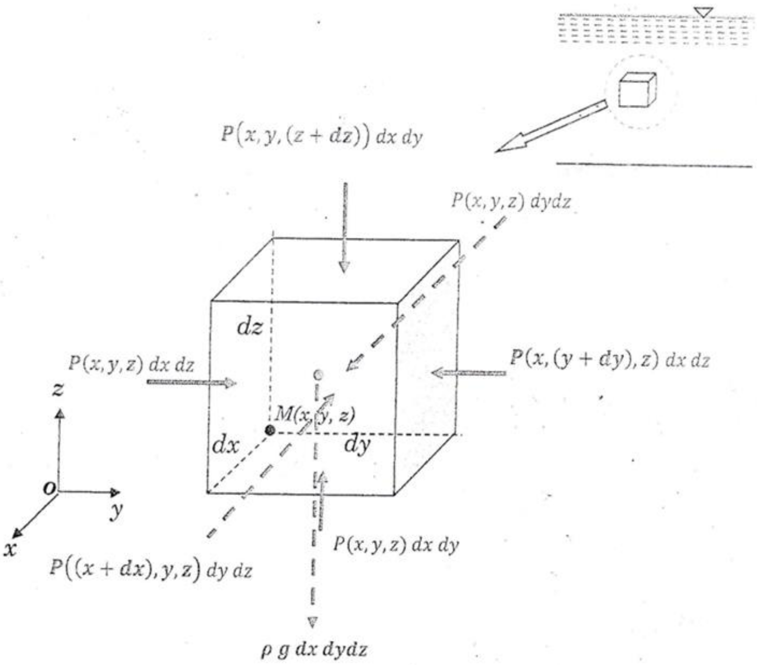


Figure II. : Forces acting on a fluid element in equilibrium in the gravity field.

The resultant of the pressure forces along the axis **(oz)** is :

$$dF_z = P(x,y,z)dxdy - p(x,y,z+dz)dxdy$$

By a first order development, we have :

$P(z+dz) = p(z) + \frac{\partial P}{\partial z}dz + \cdots$ Hence: $dF_z = -\frac{\partial P}{\partial z}dxdydz.$

Eventually ; the surface force then boils down to :

$$\overrightarrow{dF} = -\left(\frac{\partial P}{\partial x}\vec{\imath} + \frac{\partial P}{\partial y}\vec{\jmath} + \frac{\partial P}{\partial z}\vec{k}\right)dxdydz = -\left(\overrightarrow{grad}\,P\right)dV$$

The fundamental principle of statics is written:

$\sum \overrightarrow{F_{ext}} = \vec{0} \quad \Rightarrow \quad \rho dV\,\vec{g} + \vec{P}dS = \vec{0}$ Knowing that: $\{\vec{g} = -g\vec{k}\}$

$$\Rightarrow \quad -\rho g dV\,\vec{k} - \left(\frac{\partial P}{\partial x}\vec{\imath} + \frac{\partial P}{\partial y}\vec{\jmath} + \frac{\partial P}{\partial z}\vec{k}\right)dxdydz = \vec{0}$$

We finally obtain: $\frac{\partial P}{\partial x} = 0$; $\frac{\partial P}{\partial y} = 0$; $\frac{\partial P}{\partial z} = -\rho g$

This shows us that the pressure does not depend on the **x** and **y** directions; on the other hand, it only depends on direction **z.**

Hence the differential equation to solve to know the pressure at any point of the fluid at rest: $\frac{dP}{dz} = -\rho g$

The fundamental equation of statics can be established in a more general way, without involving a particular reference. $\rho\vec{g} - \overrightarrow{grad}\, P$=0 ; either :

$\overrightarrow{grad}\, P = \rho\vec{g}$, fundamental vector relation of fluid statics.

II.2 Hydrostatic pressure in an incompressible fluid

Let $\overrightarrow{dF}$ be the elementary force exerted by the particles of fluid **medium 2** on the particles of fluid **medium 1** through surface element dS. This force is broken down into two components :

- A tangential component $\overrightarrow{dF_T}$.
- A normal component $\overrightarrow{dF_N}$.

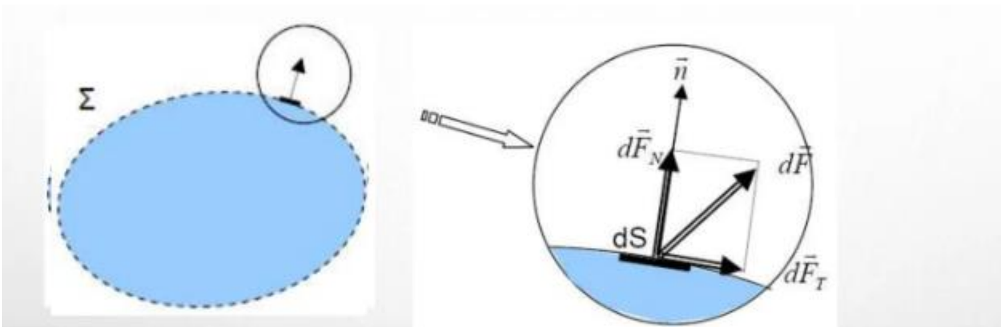


Figure II.1 the components $\overrightarrow{dF_T}$ and $\overrightarrow{dF_N}$ of the force $\overrightarrow{dF}$ on a fluid surface element.

The tangential component $\overrightarrow{dF_T}$ acting on the surface element dS is called **shear stress**, and the normal component $\overrightarrow{dF_N}$ acting on the surface element **dS** is called **normal stress**.

In fluid statics, no friction forces in the fluid, the **tangential component** cancels out, because this component linked to **viscosity,** all the forces, which develop on the surfaces, are due **to pressure (normal stress).** $P = \frac{dF_N}{dS}$

Knowing that the unit of pressure in the international system is the pascal (Pa):

$$1Pa = 1N.m^{-2} = 1Kg.m^{-1}s^{-2}$$

$$1bar = 10^5 Pa$$

1atm=760mmHg

1atm=1.013 10^5 Pa

1 atm= 10 m de colonne d'eau (mce)

1Kgf.cm^{-2} = 98070Pa.

II.2.1 Pressure at a point in a fluid

In a fluid at rest, pressure designates the force per unit area, which is exerted perpendicularly on a surface element. The pressure is a **scalar quantity**, and <u>does not depend on the orientation of the surface element</u> used to define it.

To do this, let us take any portion of a fluid at rest, as shown by **figure.**

We call **P_1, P_2, P_3** the pressures in the 3 surfaces in the plane **(ox, oz)**. The only forces which act on the volume are those of **pressure** and the **weight of the fluid**.

The equilibrium condition results in the relationship :

$$\sum \vec{F} = m\,\vec{a} = \vec{0} \quad \Rightarrow \quad \overrightarrow{F_{poids}} + \overrightarrow{F_{pression}} = \vec{0}$$

The projection of these forces according to the direction **x** is written:

$P_1 \Delta y \Delta z - P_3 \Delta y\, l \sin\theta = 0$ where: $\Delta z = l \sin\theta$

Hence : $P_1 \Delta y \Delta z - P_3 \Delta y\, \Delta z = 0 \;\Rightarrow\; \boxed{P_1 = P_3}$

The projection of these forces according to the **z** direction is written :

$P_2 \Delta x \Delta y - P_3 \Delta y\, l \cos\theta - \frac{1}{2}\rho g\, \Delta x \Delta y \Delta z = 0$ where: $\Delta x = l \cos\theta$

Hence : $P_2 \Delta x \Delta y - P_3 \Delta x\, \Delta z - \frac{1}{2}\rho g\, \Delta x \Delta y \Delta z = 0 \quad \Rightarrow \quad P_2 - P_3 - \frac{1}{2}\rho g\, \Delta z = 0.$

If we reduce the fluid volume element to a point, i.e. Δ**z**→**0**, we obtain :

$$P_2 = P_3$$

Finally, we obtain:

$$P_1 = P_2 = P_3 = P$$

This proves at a point of a fluid at rest, the pressure is the same in all directions.

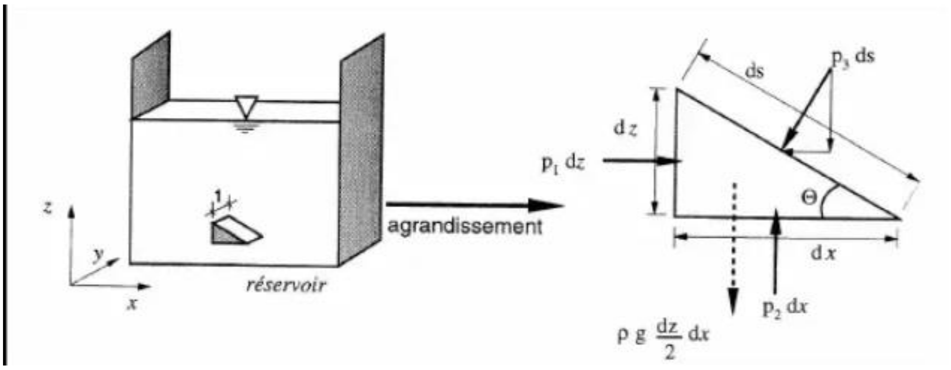


Figure II.2 Forces acting on an element of fluid in equilibrium

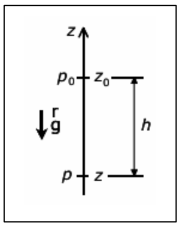
II.2.2. Application of the fundamental hydrostatic equation to the incompressible fluid

incompressible fluid, the density of the fluid is the same at every point ($\boldsymbol{\rho = cste}$);

Furthermore, we can consider that the acceleration of gravity is negligible (small variation with altitude) **g=cste**

Therefore the fundamental equation of fluid statics is:

$$\frac{dP}{dz} = -\rho g = cste$$



And by integral : $P(z) = \int dp(z) = -\rho g z + cste$

So : $\boldsymbol{P(z) + \rho g z = cste}$ ***Fundamental equation of hydrostatics***

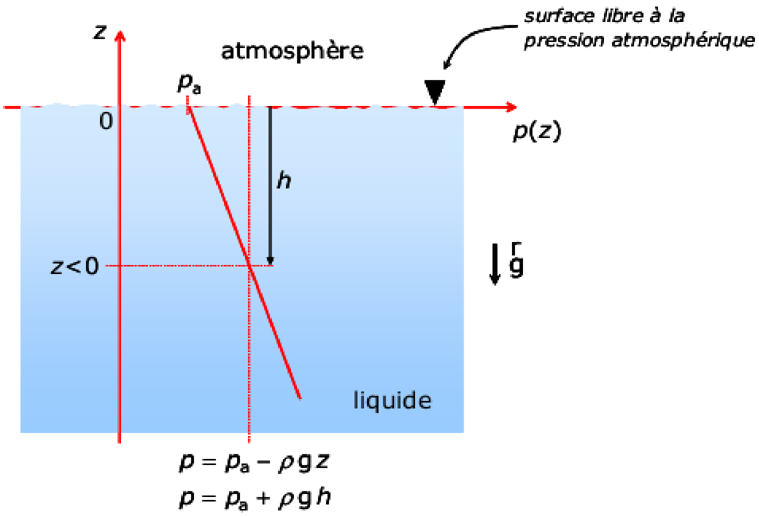
Either : $P(z) + \rho g z = P_0 + \rho g z_0$ Where P_0 is the pressure at altitude z_0

$\boldsymbol{P(z) = P_0 + \rho g(z_0 - z) = P_0 + \rho g h}$ Such a pressure field is called a **hydrostatic pressure field**.

Where **h** is the fluid height below the reference level.

Most of the time we take $z_0 = 0$ the reference level corresponding to the free surface of the fluid where$P_0 = P_{atm}$; for digital applications standard atmospheric pressure $P_{atm} = 1.013\ 10^5 Pa.$

*Knowing that **the free surface** of a liquid in equilibrium **is a flat and horizontal surface** ***(isobaric surface)***, the pressure at any point on this surface is equal to atmospheric pressure, of constant value **P=cste**.



Example :

Assuming the atmospheric pressure equal to **P_0 = 1.013 10^5 Pa**, we will determine the pressure **P** which will prevail outside a submarine, sunken and immobile, at a point located at a depth of **38 m**. The density of water is taken equal to **1 000 kg/m^3**.

According to the hydrostatic relation, we immediately have:

P = 1,013 10^5 + 1000 * 9,81 * 38 = 4,74 105 Pa.

II.2.3 Concept of absolute pressure and effective pressure

We apply the fundamental equation of hydrostatics in order to calculate the pressure at the point **M** located below the free surface. Knowing that the pressure at the free surface is equal to $\mathbf{P_{atm}}$

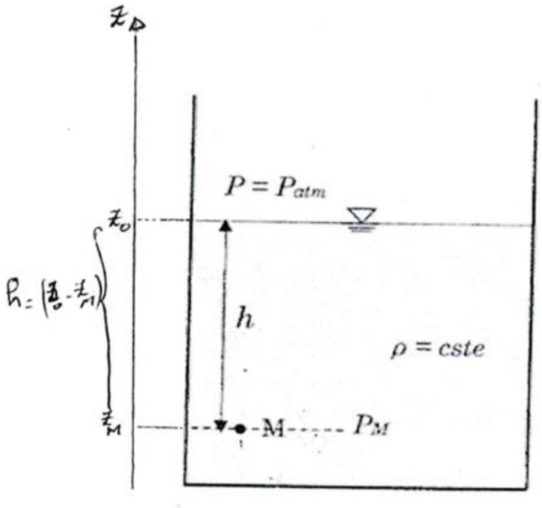


Figure II.3 One-point pressure

- **Absolute pressure** is the actual pressure at a given position, and it is measured relative to absolute vacuum (i.e. zero absolute pressure), so it is always positive.

Applying **the fundamental equation of hydrostatics** $\boldsymbol{P(z) + \rho g z = cste}$

We obtain: $P_M + \rho g Z_M = P_{atm} + \rho g Z_0$

$$\Rightarrow \quad P_M = P_{atm} + \rho g (Z_0 - Z_M) \quad \text{hence:} \quad h = (Z_0 - Z_M)$$

Z_M , Z_0 it is the altitude of point M and free surface successively.

$$\Rightarrow \quad \boldsymbol{P_M = P_{atm} + \rho g h}$$

$\boldsymbol{P_M = P_{atm} + \rho g h}$ *is called* ***absolute pressure***

$$\boldsymbol{P_{absolue} = P_{atm} + \rho\, g h}$$

- **Effective pressure** (relative pressure) is the pressure that is measured in relation to atmospheric pressure. This pressure can be positive or negative.

  $P_{absolu} - P_{atm} = \rho g h$ *is called* ***effective pressure***

$$\boldsymbol{P_{effective} = P_{absolue} - P_{atm} = \rho\, g h}$$

Most pressure measuring devices indicate the effective pressure.

Note : $P_{absolue} = P_{effective} + P_{atm} = \rho\, gh$

II.2.4 Pressure for immiscible fluids

The pressure in liquid at rest increases linearly with depth.

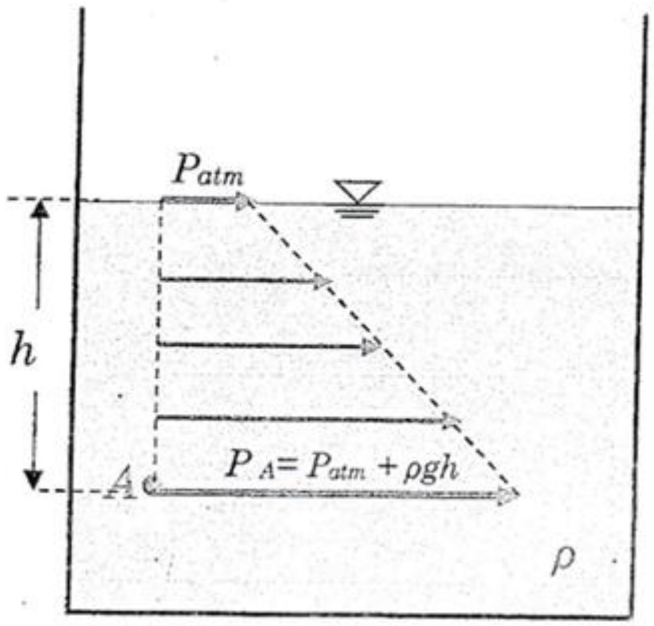


Figure II.4 variation of pressure in a liquid at rest with depth

For the diagram above and according to the fundamental equation of hydrostatics, we want to calculate $\mathbf{P_A}$ and $\mathbf{P_B}$ we have:

In the fluid 1 : $P_A = P_B - \rho_2 gh \quad \Rightarrow \quad P_A - P_B = -\rho_2 gh$ ……(1)

In the fluid 2 : $P_B = P_A + \rho_1 gh \quad \Rightarrow \quad P_A - P_B = -\rho_1 gh$ …….(2)

(1)=(2) ⇒ $\rho_2 gh = \rho_1 gh \quad \Rightarrow \quad gh(\rho_2 - \rho_1) = 0$

$hense \quad g \neq 0 \; and \; (\rho_2 - \rho_1) \neq 0 \quad \Rightarrow \quad h = 0$

Therefore the free surface of a liquid or the separation surface of two immiscible liquids in equilibrium is a ***horizontal plane***.

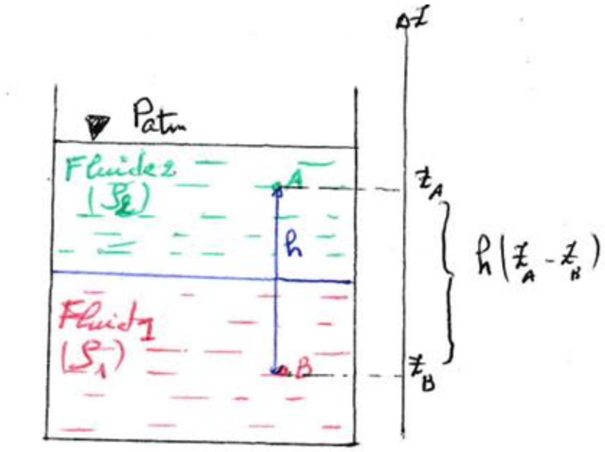


Figure II.5 the separation surface of two immiscible fluids

Now we consider two immiscible superimposed fluids, points **A** and **C** are located at **fluid 1** and **fluid 2** respectively. Point **B** belongs to **fluid 1** on the one hand and to **fluid 2** on the other hand, because it is located at the separation surface of two fluids.

The pressure at the separation surface of two immiscible fluids; we apply the fundamental equation of hydrostatics, we find:

In fluid 1 : $P_B = P_A - \rho_1 g h_1$

In fluid 2 : $P_B = P_C + \rho_2 g h_2$.

(1) =(2) Hence : $P_A - \rho_1 g h_1 = P_C + \rho_2 g h_2$

The pressure is the same on either side of the separation surface of two immiscible fluids.

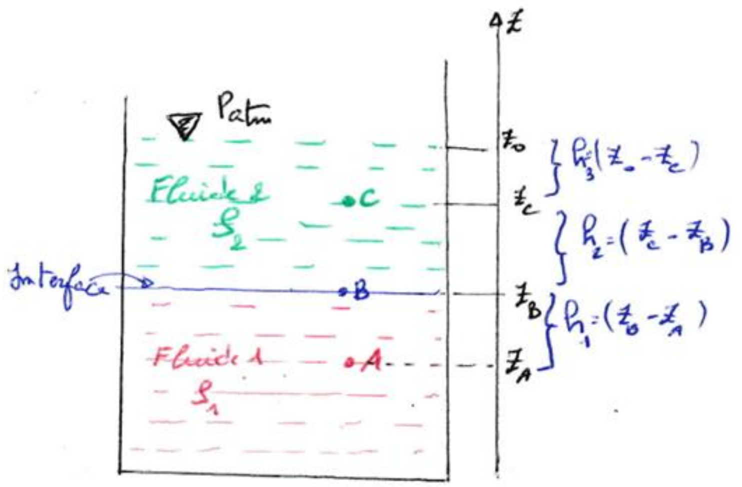


Figure II.6 interface between two immiscible fluids

- If we want to calculate the pressure at point **A** ($\boldsymbol{P_A}$), By applying the hydrostatic equation, we find: $P_c = P_{atm} + \rho_2 g h_3$

$$P_A = P_{atm} + \rho_1 g h_1 + \rho_2 g h_3 + \rho_2 g h_2$$

II.3 Static of a compressible fluid

The fundamental equation of static fluids is given by :

$$\frac{dP}{dz} = -\rho g$$

The density of the compressible fluid depends on the temperature or the pressure we must then know the function $\boldsymbol{\rho(P,T)}$.

To simplify the study, we will take the case of an ideal gas :

From the equation of state of an ideal gas write (see chapter 1) :

$$\boldsymbol{\rho = \frac{P}{RT} \quad avec\ R = \frac{r}{M_{gaz}}} \qquad \textbf{(*)}$$

Or :

P : Absolute pressure

T : absolute temperature

R : specific constant of a gas

M_{gaz} : molar mass of the gas

r: universal constant of ideal gases

By introducing $\boldsymbol{\rho}$ into the relation (*) we find:

$$\frac{dP}{dz} = -\rho g = -g\frac{\mathrm{P}}{\mathrm{RT}}$$

We assume that the fluid is **isothermal** ***(T does not vary with z)***, and we also assume that the acceleration of gravity varies weakly with z.

By integration, we obtain:

$$\int_{P_0}^{P}\frac{dP}{P} = -\frac{g}{RT}\int_{z_0}^{z} dz \quad \Longrightarrow \quad ln\frac{P}{P_0} = -\frac{g}{RT}(z - z_0)$$

Or again $\boldsymbol{P(z) = P_0 exp(-\frac{g}{RT}(z - z_0)}$ It is called ***the fundamental equation of aerostatics***.