Rappel Mathemetics

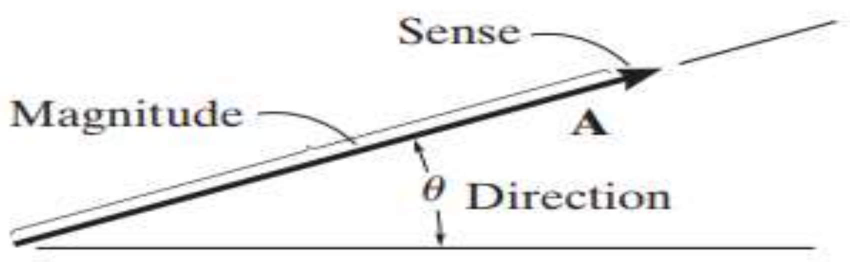
Vectors and scalars

All physical quantities in engineering mechanics are measured using either scalars or vectors.

Scalar. A *scalar* is any positive or negative physical quantity that can be completely specified by its *magnitude*. Examples of scalar quantities include length, mass, and time.

Vector. A *vector* is any physical quantity that requires both a *magnitude* and a *direction* for its complete description. Examples of vectors encountered in statics are force, position, and moment. A vector is shown graphically by an arrow. The length of the arrow represents the *magnitude* of the vector, and the angle between the vector and a fixed axis defines the *direction of its line of action*. The head or tip of the arrow indicates the *sense of direction* of the vector, Fig.1.

In print, vector quantities are represented by bold face letters such as **A**, and its magnitude of the vector is italicized, *A*. For handwritten work,
it is often convenient to denote a vector quantity by simply drawing an
arrow on top of it, A:.

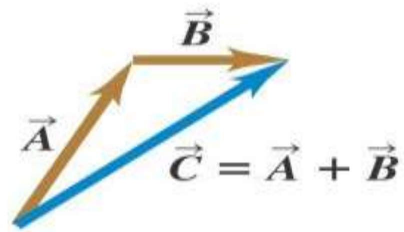


Vectors representing physical quantities can be classified as free, sliding, or fixed.
A ***free vector*** is one whose action is not confined to or associated with a unique line in space. For example, if a body moves without rotation, then the movement or displacement of any point in the body may be taken as a vector. This vector describes equally well the direction and magnitude of the displacement of every point in the body. Thus, we may represent the displacement of such a body by a free vector.
A ***sliding vector*** has a unique line of action in space but not a unique point of application. For example, when an external force acts on a rigid body, the force can be applied at any point along its line of action without changing its effect on the body as a whole, and thus it is a sliding vector.
A ***fixed vector*** is one for which a unique point of application is specified. The action of a force on a deformable or nonrigid body must be specified by a fixed vector at the point of application of the force. In this instance the forces and deformations within the body depend on the point of application of the force, as well as on its magnitude and line of action.

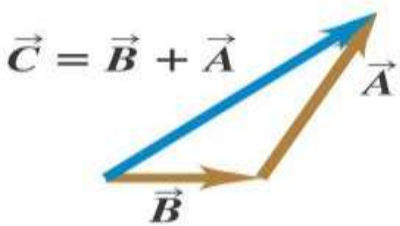
Adding two vectors graphically

Two vectors may be added graphically using either the parallelogram method or the head-to-tail method.

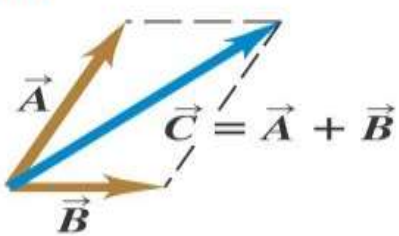
(a) We can add two vectors by placing them head to tail.



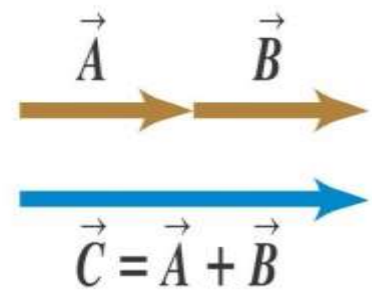
(b) Adding them in reverse order gives the same result.



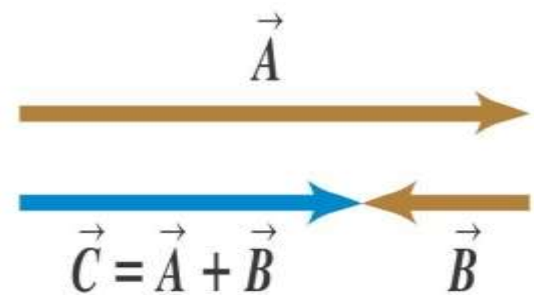
(c) We can also add them by constructing a parallelogram.



(a) The sum of two parallel vectors



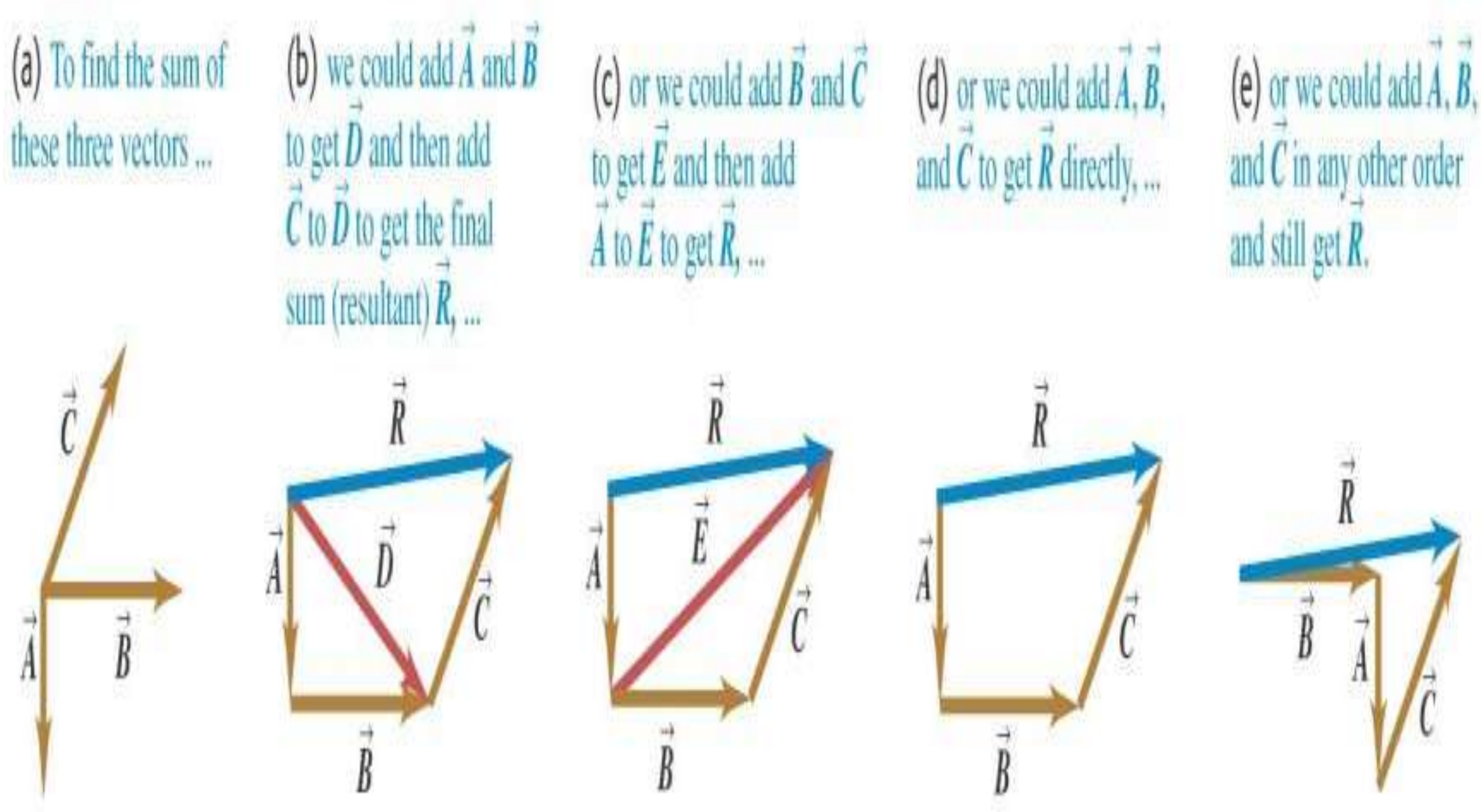
(b) The sum of two antiparallel vectors



Adding more than two vectors graphically

To add several vectors, use the head-to-tail method.

- The vectors can be added in any order.



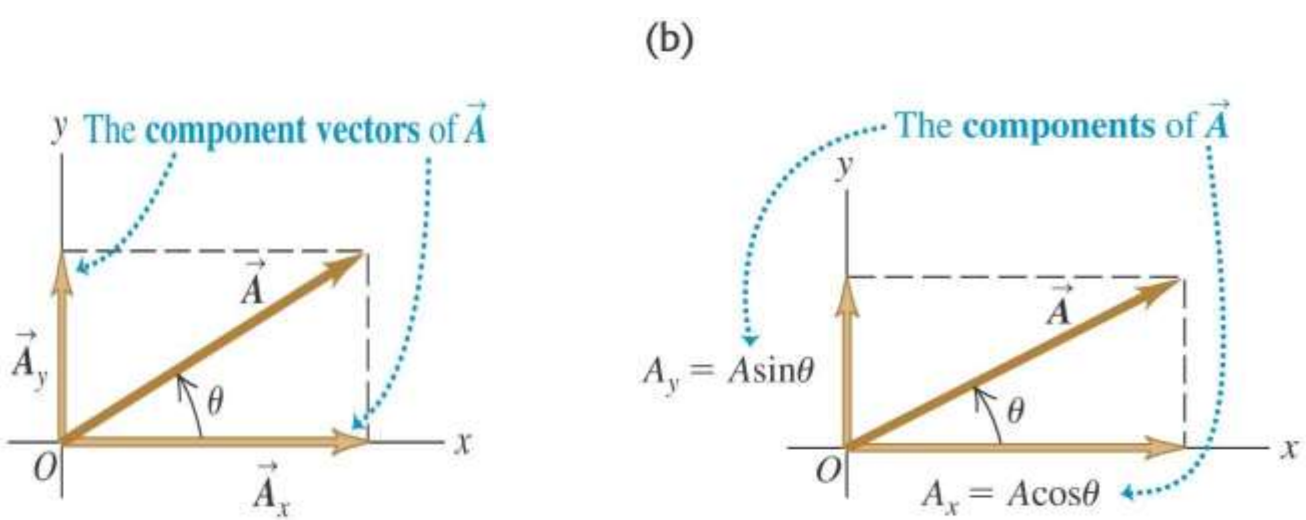
Components of a vector

- Adding vectors graphically provides limited accuracy. Vector components provide a general method for adding vectors.
- Any vector can be represented by an x component A_x and a y-component A_y.
- Use trigonometry to find the components of a vector:

$A_x = A\cos\theta$ and $A_y = A\sin\theta$, where θ is measured from the +x-axis toward the +y-axis.

Components of a vector

- Adding vectors graphically provides limited accuracy. Vector components provide a general method for adding vectors.
- Any vector can be represented by an x-component Ax and a y-component Ay.
- Use trigonometry to find the components of a vector:
- Ax = Acos θ and Ay = Asin θ, where θ is measured from the +x-axis toward the +y-axis.



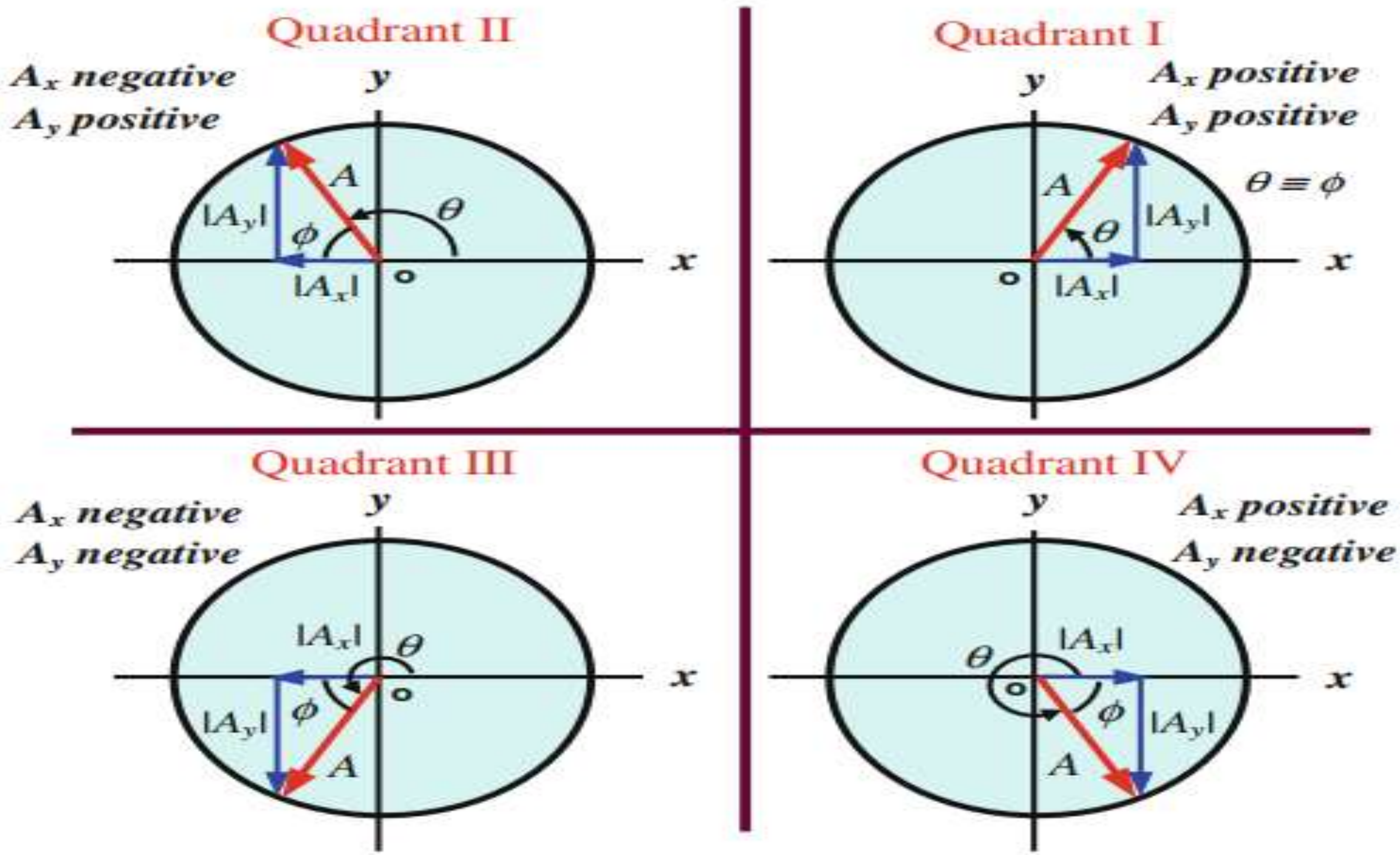
The magnitudes *Ax* and *Ay* form two sides of a right triangle that has a hypotenuse of magnitude *A.* Thus, from *Ax* and *Ay* we get:

$$A = \sqrt{A_x^2 + A_y^2}$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

Positive and negative components

- The components of a vector can
- be positive or negative numbers,
- as shown in the figure.



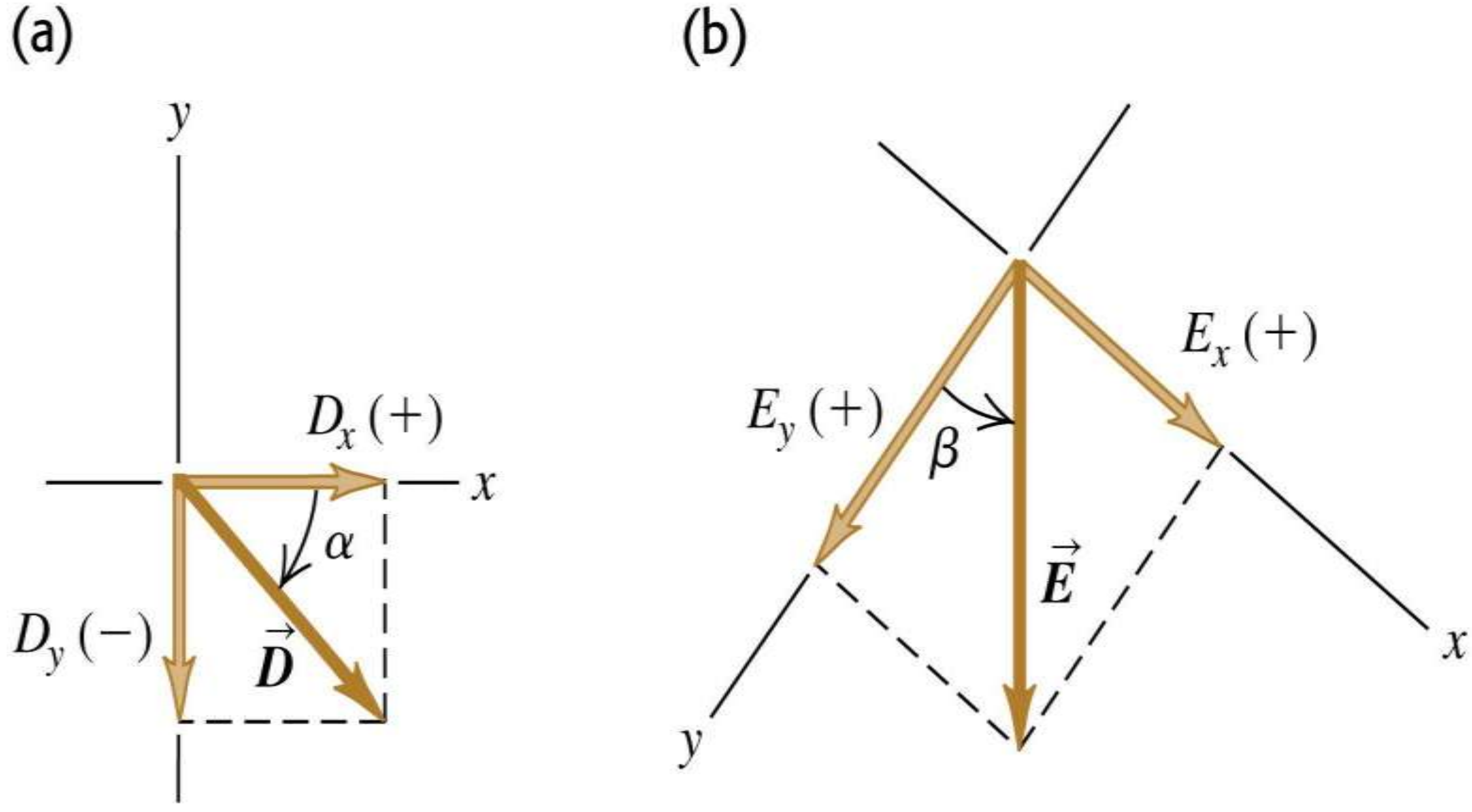
The signs of A_x and A_y depend on the quadrant where the vector $\vec{A}$ is located

Table 1. Calculating θ from φ according to the signs of A_x and A_y

Sign of A_x	Sign of A_y	Quadrant	Angle θ
$+$	$+$	I	$\theta = \phi$
$-$	$+$	II	$\theta = 180° - \phi$
$-$	$-$	III	$\theta = 180° + \phi$
$+$	$-$	IV	$\theta = 360° - \phi$

Finding components

- We can calculate the components of a vector from its magnitude and direction.



Calculations using components

- We can use the components of a vector to find its magnitude and direction:
- We can use the components of a set of vectors to find the components of their sum:
- Refer to Problem-Solving
- Strategy

Magnitude of a vector using components

This method can be generalized to three-dimensional vectors as:

$A = A_x \mathrm{i} + A_y \mathrm{j} + A_z \mathrm{k}$

$$\begin{cases} \vec{OA}\begin{cases} X_A = OA_x \\ Y_A = OA_y \\ Z_A = OA_z \end{cases} \\ \vec{OA} = \vec{X}_A.\vec{i} + \vec{Y}_A.\vec{j} + \vec{Z}_A.\vec{k} \end{cases}$$

Représentation d'une distance

$$\vec{OA}\begin{pmatrix} X_A \\ Y_A \\ Z_A \end{pmatrix} \vec{OB}\begin{pmatrix} X_B \\ Y_B \\ Z_B \end{pmatrix}$$

$$\vec{AB} = \vec{AO} + \vec{OB} = \vec{OB} - \vec{OA}$$

$$\vec{AB} = (\vec{X}_B - \vec{X}_A).\vec{i} + (\vec{Y}_B - \vec{Y}_A).\vec{j} + (\vec{Z}_B - \vec{Z}_A).\vec{k}$$

$$AB = \sqrt{(X_B - X_A)^2 + (Y_B - Y_A)^2 + (Z_B - Z_A)}^2$$

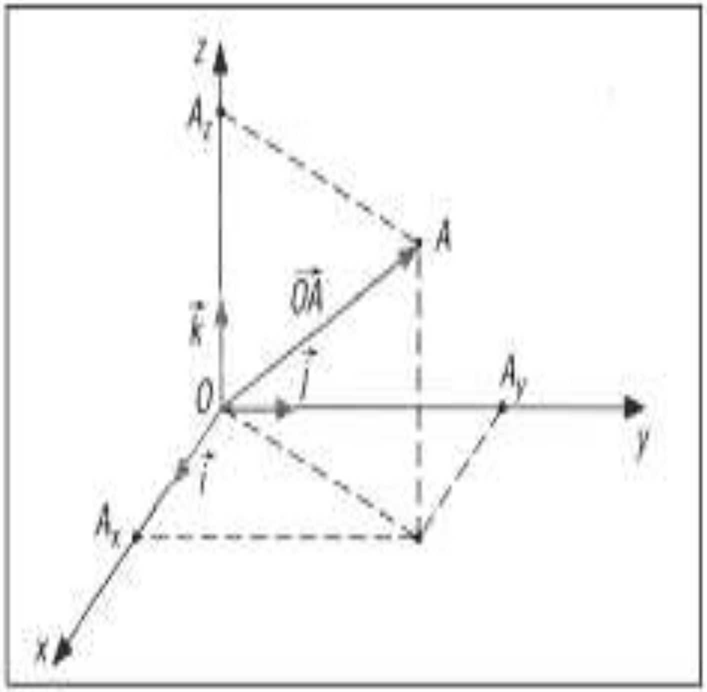
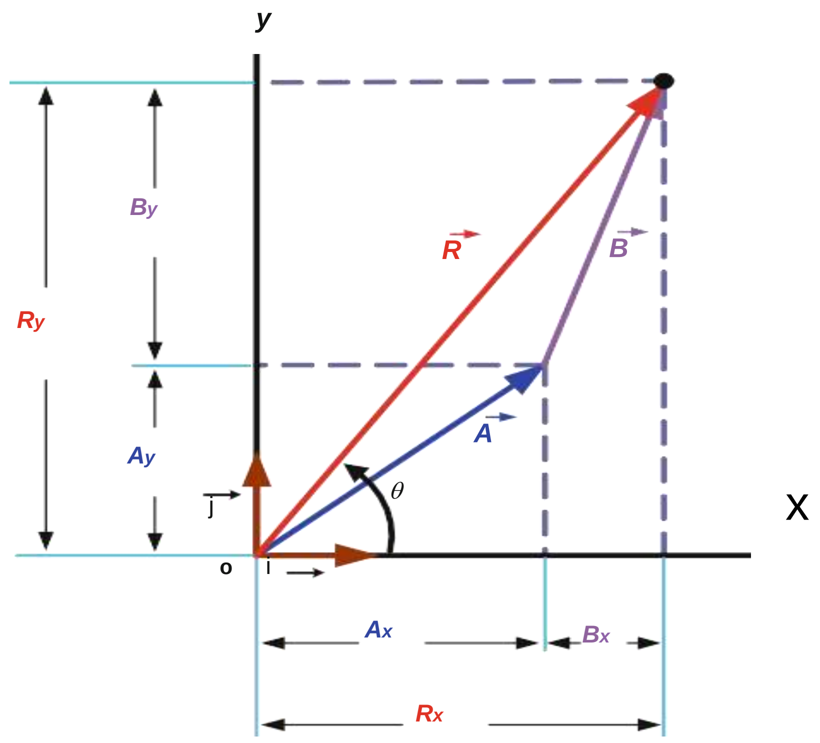


Figure 1.: Un point dans l'espace.

Fig. 2. Geometric representation of the sum of the two vectors A and B , showing the relationship between the components of the resultant R and the components of A and B.

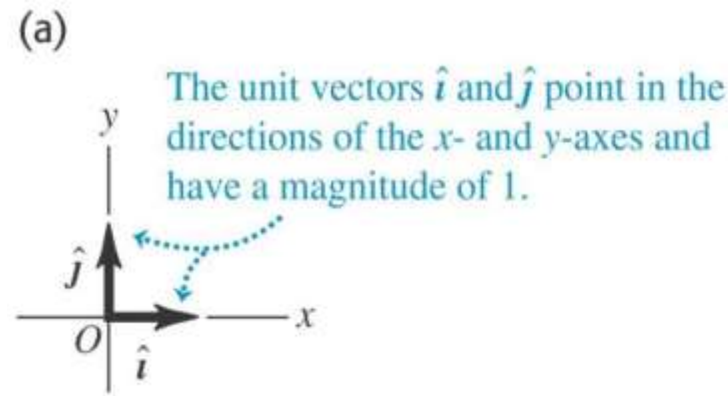


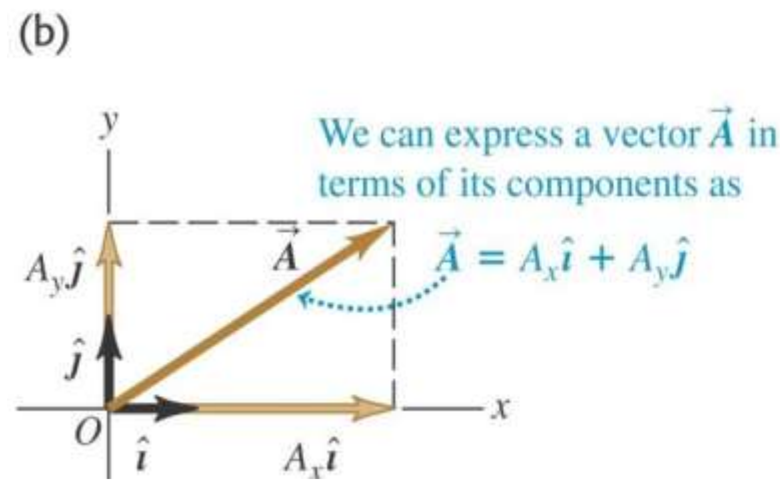
Unit vectors

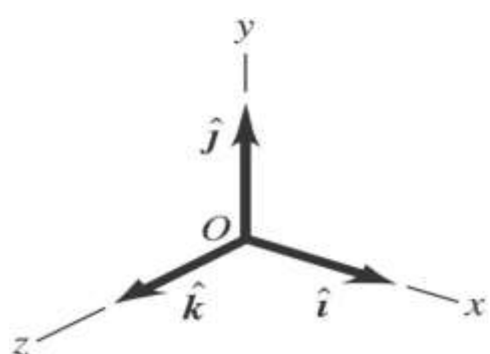
- A unit vector has a magnitude of 1 with no units.
- The unit vector î points in the +x-direction, points in the +y-direction, and points in the +z-direction.
- Any vector can be expressed in terms of its components as

$A = A_x \hat{i} + A_y\, j + A_z\, k$.

- Follow Example 1.

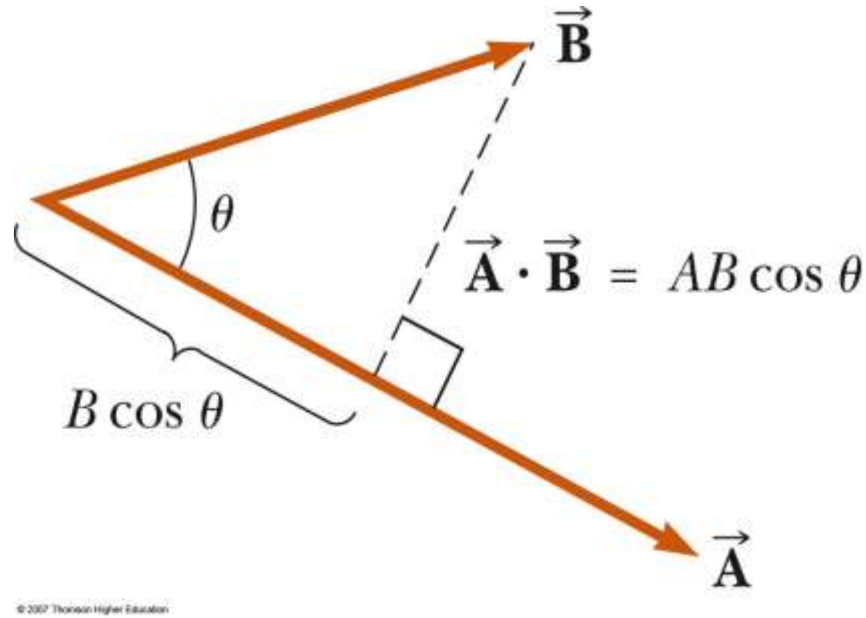






The scalar product

- The scalar product of two vectors is written as $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$
  - It is also called the dot product
- $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} \equiv A\, B\, cos\; \theta$
  - *q* is the angle *between A* and *B*
- Applied to work, this means



$$W = F\Delta r\cos\theta = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{r}}$$

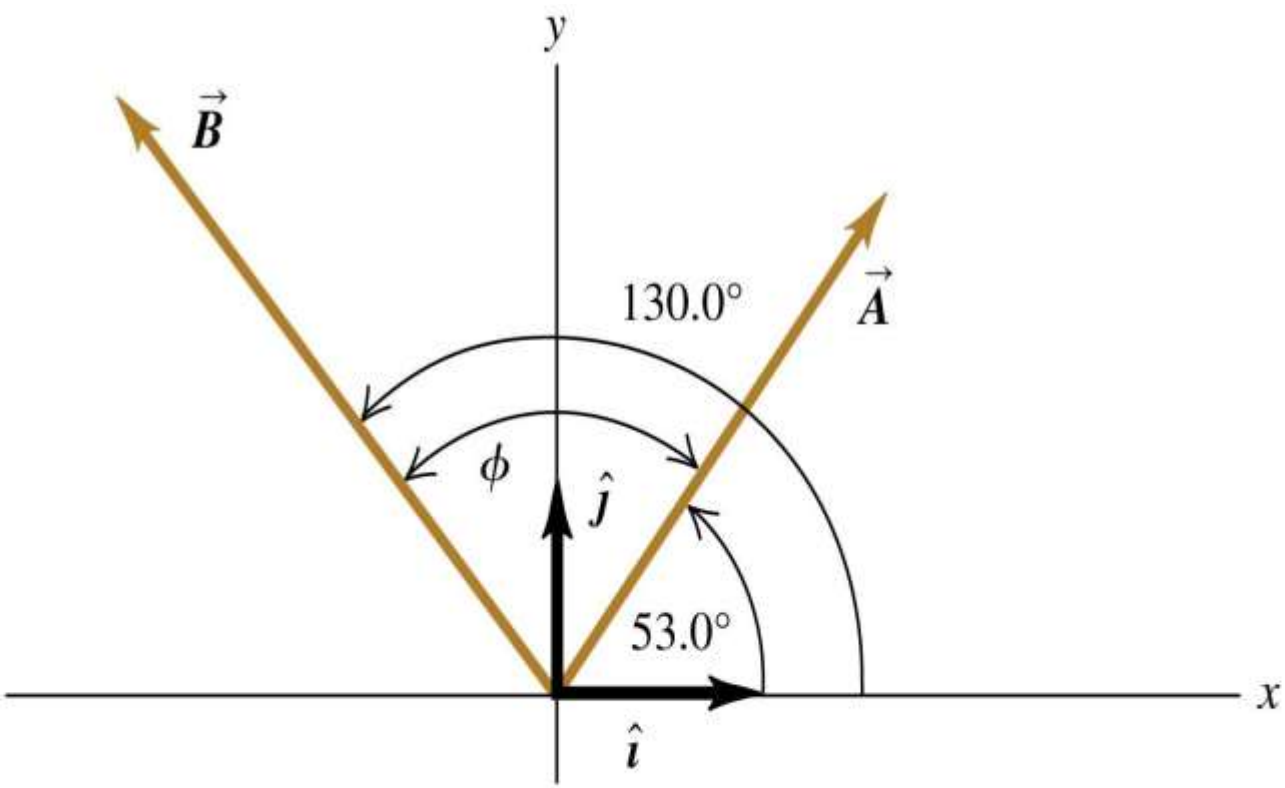
Derivation

- How do we show that $\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$
- Start with
$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$
- Then
$$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_x\hat{i}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_y\hat{j}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_z\hat{k}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})\end{aligned}$$
- But
$$\hat{i}\cdot\hat{j} = 0;\ \ \hat{i}\cdot\hat{k} = 0;\ \ \hat{j}\cdot\hat{k} = 0$$
$$\hat{i}\cdot\hat{i} = 1;\ \ \hat{j}\cdot\hat{j} = 1;\ \ \hat{k}\cdot\hat{k} = 1$$
- So
$$\begin{aligned}\vec{A}\cdot\vec{B} &= A_x\hat{i}\cdot B_x\hat{i} + A_y\hat{j}\cdot B_y\hat{j} + A_z\hat{k}\cdot B_z\hat{k} \\ &= A_x B_x + A_y B_y + A_z B_z\end{aligned}$$

Calculating a scalar product

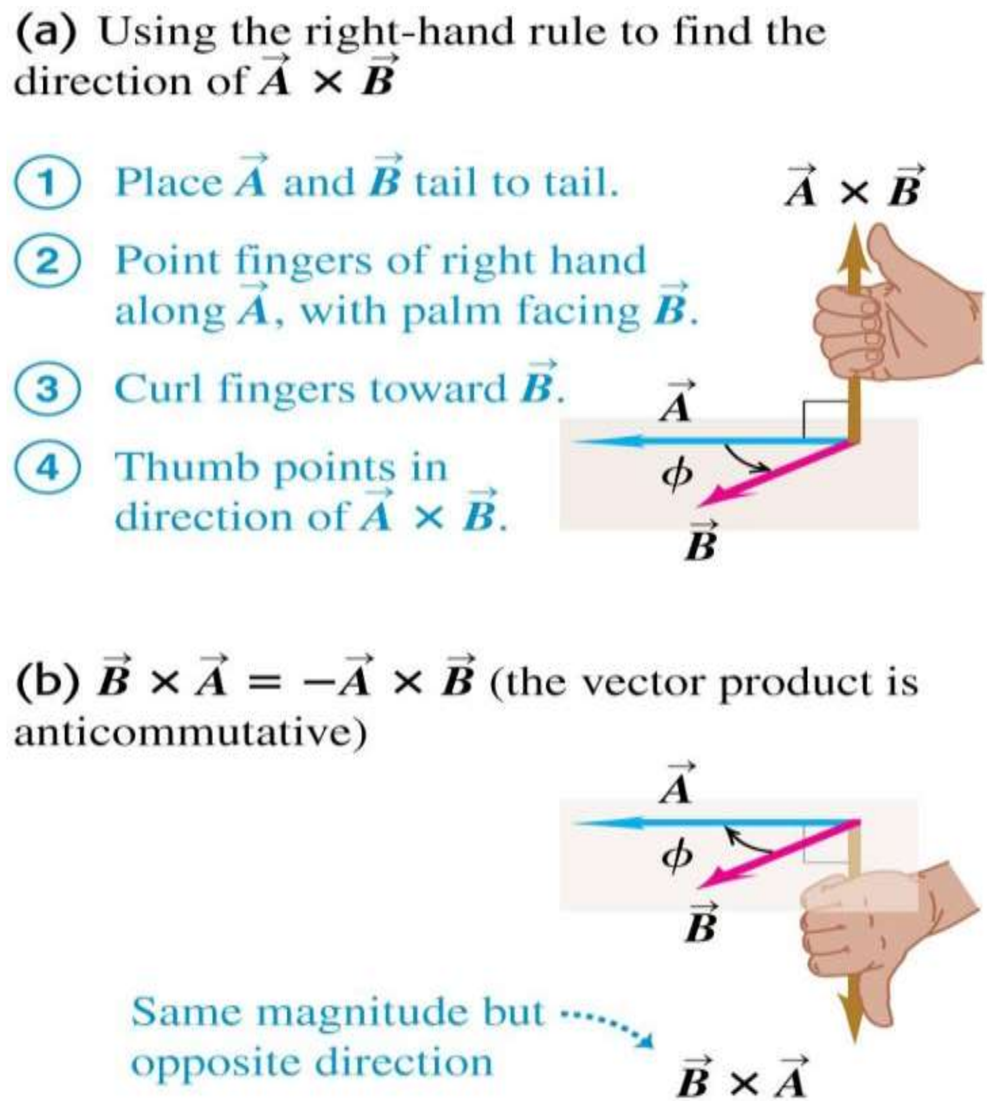
In terms of components,

- Example 1 shows how to calculate a scalar product in two ways.



The vector product

The vector product ("cross product") of two vectors has magnitude and the right-hand rule gives its direction.



$$\vec{A} \wedge \vec{B} = \vec{A} \times \vec{B} = \vec{C} = \left\|\vec{A}\right\| . \left\|\vec{B}\right\| . \sin(\theta) . \vec{u}_c$$

;

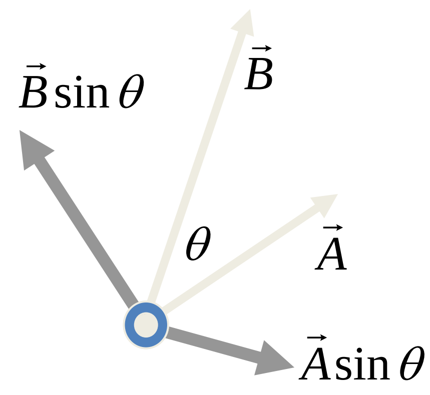
$$\vec{A} = \begin{cases} A_x \\ A_y \\ A_z \end{cases} \qquad \vec{B} = \begin{cases} B_x \\ B_y \\ B_z \end{cases}$$

$$\vec{A} \wedge \vec{B} = \begin{cases} A_x \\ A_y \\ A_z \end{cases} \wedge \begin{cases} B_x \\ B_y \\ B_z \end{cases} = \begin{vmatrix} A_y & B_y \\ A_z & B_z \end{vmatrix} \vec{i} - \begin{vmatrix} A_x & B_x \\ A_z & B_z \end{vmatrix} \vec{j} + \begin{vmatrix} A_x & B_x \\ A_y & B_y \end{vmatrix} \vec{k} = \begin{cases} A_y . B_z - A_z . B_y \\ A_z . B_x - A_x . B_z \\ A_x . B_y - A_y . B_x \end{cases}$$

$$\vec{A} \wedge \vec{B} = (A_y . B_z - A_z . B_y) \vec{i} + (A_z . B_x - A_x . B_z) \vec{j} + (A_x . B_y - A_y . B_x) \vec{k}$$

Cross Product

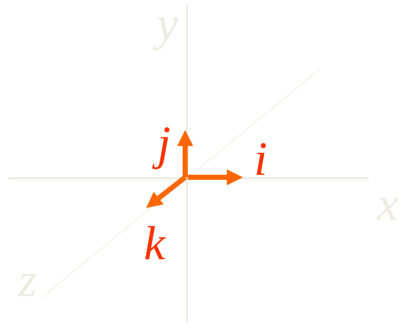
$$\vec{C} = \vec{A} \times \vec{B}$$



The cross product of two vectors says something about how perpendicular they are.

Magnitude: $$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = AB\sin\theta$$

θ is smaller angle between the vectors
Cross product of any parallel vectors = zero
Cross product is maximum for perpendicular vectors
Cross products of Cartesian unit vectors:

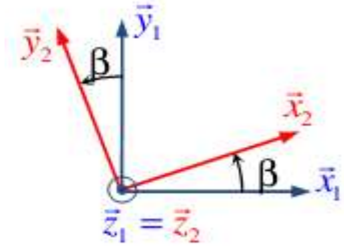


$$\hat{i} \times \hat{j} = \hat{k};\ \ \hat{i} \times \hat{k} = -\hat{j};\ \ \hat{j} \times \hat{k} = \hat{i}$$

$$\hat{i} \times \hat{i} = 0;\ \ \hat{j} \times \hat{j} = 0;\ \ \hat{k} \times \hat{k} = 0$$

- Change of base for a vector, using projections

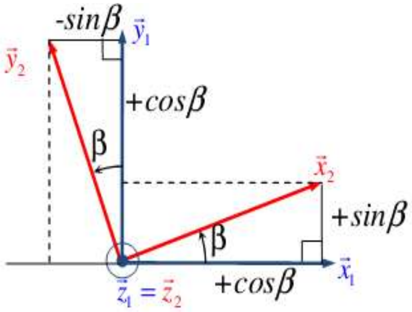
We define the base $R_1(\vec{x}_1, \vec{y}_1, \vec{z}_1)$ and the base $R_2(\vec{x}_2, \vec{y}_2, \vec{z}_2)$ as follows:



$$\vec{x}_2 = \cos(\beta).\vec{x}_1 + \sin(\beta).\vec{y}_1$$

$$\vec{y}_2 = -\sin(\beta).\vec{x}_1 + \cos(\beta).\vec{y}_1$$

$$\vec{z}_2 = \vec{z}_1$$



$$\vec{x}_1 = \cos(\beta).\vec{x}_2 - \sin(\beta).\vec{y}_2$$
$$\vec{y}_1 = \sin(\beta).\vec{x}_2 + \cos(\beta).\vec{y}_2$$
$$\vec{z}_1 = \vec{z}_2$$