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Normed Vector Spaces
Exercices of chapter 1 :Banach spaces

Exercise1 Let be E a normed vector space and $(x_n)_{n\infty\mathbb{N}}$ a sequence of elements of E. Supose that (x_n) is a Cauchy sequence. Show that it converges if and only if it has a convergent subsequence .

Exercise 2 Let be $X =]0, \infty[$. For $x, y \in X$, note

$$\delta(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|$$

Show that δ is a distance on X. Is the metric space (X, d) complet?

Exercise 3 Let bet E the vectorial space of $\mathbb{R}$ valued continuous functions on $[-1, 1]$ .Define a norm on E by

$$\|f\|_1 = \int_{-1}^{1} |f(t)|\, dt$$

We want to show that E endowed with this norm is not complet. To show that we define a sequence of functions $(f_n)_{n\in\mathbb{N}^*}$ by

$$f_n(t) = \begin{cases} -1 \text{ if } -1 \leq t \leq -\frac{1}{n} \\ nt \text{ if} -\frac{1}{n} \leq t \leq \frac{1}{n} \\ 1 \text{ if } \quad \frac{1}{n} \leq t \leq 1 \end{cases}$$

1- verify that $f_n \in E$, $\forall n \geq 1$.
2- Show that

$$\|f_n - f_p\| \leq \sup\left(\frac{2}{n}, \frac{2}{p}\right)$$

and deduce that $(f_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence.
3- Supose that there exists a function $f \in E$ so that (f_n) converges to f in $(E, \|\|_1)$. Then show that we have:

$$\lim_{n\to+\infty} \int_{-1}^{-\alpha} |f_n(t) - f(t)|\, dt = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \int_{\alpha}^{1} |f_n(t) - f(t)|\, dt = 0$$

for all $0 < \alpha < 1$.

4- therefore show that

$$\lim_{n\to+\infty} \int_{-1}^{-\alpha} |f_n(t) + 1|\, dt = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \int_{\alpha}^{1} |f_n(t) - 1|\, dt = 0$$

for all $0 < \alpha < 1$. Deduce that

$$\begin{aligned} f(t) &= 1 \text{ for all } -1 \leq t < 0 \\ f(t) &= -1 \text{ for all } 0 < t \leq 1 \end{aligned}$$

Conclude.

Exercise4

Let X be a Banach space, Y a normed vectorial space and $T : X \to Y$ a continuous lineare mapping. Supose that there exists a constante $c > 0$ so that:

$$\|Tx\| \geq c\,\|x\| \quad \text{for all } x \in X.$$

1- Show that $\mathrm{Im}(T)$ is closed in Y.

2- Show that T is an isomorphism from X to $\mathrm{Im}(T)$.

Exercise 5 Let be E the vectorial space of $\mathbb{C}$ valued continuous functions on $[-1, 1]$, endowed with the norm *sup* : $\|f\|_\infty = \underset{t\in[-1,1]}{Sup}\,|f(t)|$

Let be F the vectorial space of $2\pi-$périodique continuous functions on $\mathbb{R}$, endowed with the norms N_2 so that $N_2(f) = \frac{1}{2\pi}\sqrt{\int_{-\pi}^{\pi} |f(t)|^2\,dt}$, or the norm *sup* $N_\infty : N_\infty(f) = \underset{t\in\mathbb{R}}{Sup}\ f(t)$.

Let be $L : E \to F$ the mapping defined by $L(f)(t) = f(\cos t)$.

1- Show that L is well defined, is lineare and injective.

2- Show that L is continuous for both of the norms N_2 and N_∞ of F, and calculate for both of them, $\|L\|_2$ and $\|L\|_\infty$.