Solutions of exercises on Hypothesis Testing

Correction exercice 1. *We choose a threshold* $c \in \mathbb{N}$ *and compare the observed value of* $X = x$ *to* c. *We accept* H_0 *if* $x \leq c$ *and reject it if* $x > c$. *The probability of type I error is given by*

$$P(\textit{type I error}) = P(\textit{Reject } H_0|H_0) = P(\textit{Reject } H_0|\theta = 0.5) = P(X > c|\theta = 0.5) = \sum_{k=c+1}^{\infty} P(X = k)$$

(where $X \sim \textit{Geometric}(\theta_0 = 0.5)$*)*

$$= \sum_{k=c+1}^{\infty} (1-\theta_0)^{k-1}\theta_0 = (1-\theta_0)^c \theta_0 \sum_{l=0}^{\infty} (1-\theta_0)^l = (1-\theta_0)^c.$$

To have $\alpha = 0.05$, *we need to choose* c *such that* $(1-\theta_0)^c \leq \alpha = 0.05$, *so we obtain*

$$c \geq \frac{\ln \alpha}{\ln(1-\theta_0)} = \frac{\ln(0.05)}{\ln(0.5)} \approx 4.32.$$

Since we would like $c \in \mathbb{N}$, *we can let* $c = 5$. *To summarize, we have the following decision rule: Accept* H_0 *if the observed value of* X *is in the set* $A = \{1, 2, 3, 4, 5\}$, *and reject* H_0 *otherwise. Since the alternative hypothesis* H_1 *is a simple hypothesis (*$\theta = \theta_1$*), there is only one value for* β,

$$\beta = P(\textit{type II error}) = P(\textit{accept } H_0|H_1) = P(X \leq c|H_1) = 1-(1-\theta_1)^c = 1-(0.9)^5 = 0.41$$

Correction exercice 2. *We have a sample from a normal distribution with known variance, so using the first row in Table 8.2, we define the test statistic as*

$$W = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}.$$

We have $\bar{X} = 2.85$, $\mu_0 = 2$, $\sigma = 1$, *and* $n = 4$. *So, we obtain*

$$W = \frac{2.85 - 2}{1/\sqrt{4}} = 1.7.$$

Here, $\alpha = 0.1$, *so* $z_{\alpha/2} = z_{0.05} = 1.645$. *Since* $|W| > z_{\alpha/2}$, *we reject* H_0 *and accept* H_1. *Here, the test statistic* W *is* $W \sim 2(\bar{X} - 2)$. *If* $X \sim N(\mu, 1)$, *then* $\bar{X} \sim N(\mu, 1/4)$, *and* $W \sim N(2(\mu - 2), 1)$. *Thus, we have*

$$\beta = P(\textit{type II error}) = P(\textit{accept } H_0|\mu) = P(|W| < z_{\alpha/2}|\mu) = P(|W| < z_{\alpha/2})$$

(when $W \sim N(2(\mu-2),1)$)

$$= \Phi(z_{\alpha/2} - 2\mu + 4) - \Phi(-z_{\alpha/2} - 2\mu + 4).$$

Correction exercice 3. *Here, we have a non-normal sample, where $n = 100$ is large. Using the results of Table 8.3, specifically the second row, we define the test statistic as*

$$W = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{21.32 - 20}{\sqrt{27.6}/\sqrt{100}} = 2.51.$$

Here, $\alpha = 0.05$, so $z_\alpha = z_{0.05} = 1.645$. Since $W > z_\alpha$, we reject H_0 and accept H_1.

Correction exercice 4. *Here, we have a sample from a normal distribution with unknown mean and unknown variance. Using the third row in Table 8.4, we define the test statistic as*

$$W = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}.$$

Using the data we obtain $\bar{X} = 8.26, S = 5.10$. Therefore, we obtain

$$W = \frac{8.26 - 10}{5.10/\sqrt{4}} = -0.68.$$

Here, $\alpha = 0.05$, so $n = 4$, $t_{\alpha,n-1} = t_{0.05,3} = 2.35$. Since $W > -t_{\alpha,n-1}$, we fail to reject H_0, so we accept H_0.

Correction exercice 5. *Here, we have a non-normal sample, where $n = 81$ is large. Using the results of Table 8.4, specifically the second row, we define the test statistic as*

$$W = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{8.25 - 9}{\sqrt{14.6}/\sqrt{81}} = -1.767.$$

The P-value is P(type I error) when the test threshold c is chosen to be $c = -1.767$. Since the threshold for this test (as indicated by Table 8.4) is $-z_\alpha$, we obtain $-z_\alpha = -1.767$. Noting that by definition $z_\alpha = \Phi^{-1}(1-\alpha)$, we obtain P(type I error) as

$$\alpha = 1 - \Phi(1.767) \approx 0.0386.$$

Therefore,

$$P\text{-value} \approx 0.0386.$$