Abdelhafid Boussouf University Center, Mila
Institute of Mathematics and Computer Science
Department of Computer Science – 1st Year (2024/2025)

Algebra 2 : Final Exam

Exercise No. 1 : (6pts)

1. Let $F_1 = \left\{(x, y, z) \in \mathbb{R}^3 : z = 0\right\}$ be a subset of $\mathbb{R}^3$.
   (a) Prove that F_1 is a vector subspace of $\mathbb{R}^3$ over $\mathbb{R}$.
   (b) Determine a basis B_1 for F_1, and deduce its dimension, $\dim_{\mathbb{R}}(F_1)$.

2. Let $F_2 = \left\{(x, y, z) \in \mathbb{R}^3 : x = y = 0\right\}$, a vector subspace of $\mathbb{R}^3$ over $\mathbb{R}$.
   (a) Show that $\mathbb{R}^3 = F_1 \oplus F_2$, and deduce the dimension $\dim_{\mathbb{R}}(F_2)$ of F_2.

Exercise No. 2 : (6pts) Consider the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$.

1. Compute the matrices A^2, A^3. Then, deduce the matrix A^n for all integers $n \geq 3$.
2. For any real number $x \in \mathbb{R}$, define the matrix $M(x) = I_3 + xA + \frac{x^2}{2}A^2$.
(a) Write the matrix $M(x)$ explicitly.
(b) Prove that for all real numbers $x, y \in \mathbb{R} : M(x) \cdot M(y) = M(x + y)$.
(c) Find a real number $x' \in \mathbb{R}$ such that $M(x) \cdot M(x') = I_3$. Deduce that $M(x)$ is invertible and determine its inverse $M(x)^{-1}$.
(d) Compute the determinant $\det(M(x))$ of the matrix $M(x)$.

Exercise No. 3 : (8pts) Let $\mathbb{R}_2[X] = \left\{P = a_0 + a_1X + a_2X^2 : a_0, a_1, a_2 \in \mathbb{R}\right\}$ be the vector space of polynomials with real coefficients of degree less than or equal to 2. Let $B = \left\{P_0 = 1, P_1 = X, P_2 = X^2\right\}$ be the canonical basis of $\mathbb{R}_2[X]$. Consider the linear mapping

$$f : \begin{array}{ccc} \mathbb{R}_2[X] & \longrightarrow & \mathbb{R}_2[X] \\ P & \longmapsto & f(P) = P' - (X+1)P'' \end{array}$$

where P' and P'' denote the first and second derivatives of P, respectively.
1. Find a basis for $\ker(f)$, the kernel of f, and a basis for $Im(f)$, the image of f, and compute the dimensions $\dim(\ker(f))$ and $\dim(Im(f))$.
2. Find the matrix $A = \underset{B}{Mat}(f)$ of f with respect to the basis $B = \left\{P_0 = 1, P_1 = X, P_2 = X^2\right\}$.
3. Let $B' = \{Q_0 = X - 1, Q_1 = X^2 - 1, Q_2 = X^2 + 1\}$ be a new basis of $\mathbb{R}_2[X]$.
(a) Compute the change-of-basis matrix $H = \underset{B \longrightarrow B'}{P}$ from B to B'.
(b) Let $P = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]$. Find the scalars $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ such that $P = \lambda_0Q_0 + \lambda_1Q_1 + \lambda_2Q_2$, and deduce the coordinates of P_0, P_1, P_2 in the basis B'.
(c) Determine the matrix $H^{-1} = \underset{B' \longrightarrow B}{P}$ the change-of-basis matrix from B' to B. Deduce the matrix $A' = \underset{B'}{Mat}(f)$ of f in the new basis B'.

Department of Computer Science – 1st Year (2024/2025)
Algebra 2 : Final Exam - Answer Key
Exercise No. 1 : Solution :

1. (a) We know that

$$F_1 \text{ is a subspace of } \mathbb{R}^3 \iff$$

$$\iff \begin{cases} \text{(i) } 0_{\mathbb{R}^3} \in F_1 \ (F \neq \phi) \\ \text{(ii) } F_1 \text{ is closed under vector addition} : \forall X_1, X_2 \in F_1 : X_1 + X_2 \in F_1 \\ \text{(iii) } F_1 \text{ is closed under scalar multiplication, } \forall \lambda \in \mathbb{R}, \forall X_1 \in F_1 : \lambda \cdot X_1 \in F_1. \end{cases}$$

$$\iff \begin{cases} \text{(j) } 0_{\mathbb{R}^3} \in F_1 \\ \text{(jj) } F_1 \text{ is closed under linear combinations, } \forall X_1, X_2 \in F_1, \forall \lambda_1, \lambda_2 \in \mathbb{R} : \lambda_1 X_1 + \lambda_2 X_2 \in F_1. \end{cases}$$

(j) It is clear that $0_{\mathbb{R}^3} = (0,0,0) \in F_1$ sine $z = 0$.
(jj) Let $X_1 = (x_1, y_1, 0), X_2 \in (x_2, y_2, 0) \in F_1$ and $\lambda_1, \lambda_1 \in \mathbb{R}$. Then,

$$\lambda_1 \cdot X_1 + \lambda_2 \cdot X_2 = (\lambda_1 \cdot x_1 + \lambda_2 \cdot x_2, \lambda_1 \cdot y_1 + \lambda_2 \cdot y_2, 0) \in F_1.$$

So F_1 is a vector subspace of $\mathbb{R}^3$.
(b) To find a basis for F_1, we need to find a set of linearly independent vectors that span F_1. We have

$$F_1 = \{X = (x, y, 0) : x, y \in \mathbb{R}\} = \{X = x(1,0,0) + y(0,1,0) : x, y \in \mathbb{R}\} = Span\,\{B_1 = \{e_1 = (1,0,0), e_2 = (0,1,0)\}\}.$$

Moreover, let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\lambda_1 e_1 + \lambda_2 e_2 = 0_{\mathbb{R}^3} \Longrightarrow (\lambda_1, \lambda_2, 0) = (0,0,0)$$

$$\Longrightarrow \lambda_1 = \lambda_2 = 0$$

Hence, B_1 linearly independent and forms a basis of F_1, so $\dim(F_1) = Card(B_1) = 2$.
2. (a) We have

$$\mathbb{R}^3 = F_1 \oplus F_2 \iff \begin{cases} j) F_1 \cap F_2 = \{0_{\mathbb{R}^3}\} \\ jj) \mathbb{R}^3 = F_1 + F_2. \end{cases}$$

(j) We have

$$F_1 \cap F_2 = \left\{X = (x,y,z) \in \mathbb{R}^3 : X \in F_1 \text{ and } X \in F_2\right\} = \left\{X = (x,y,z) \in \mathbb{R}^3 : z = 0 \text{ and } x = y = 0\right\} = \{(0,0,0)\}.$$

(jj) Since $F_1 + F_2$ is a subspace of $\mathbb{R}^3$, we have

$$F_1 + F_2 \subset \mathbb{R}^3.$$

Now, let $X = (x, y, z) \in \mathbb{R}^3$, we can write

$$X = (x, y, 0) + (0, 0, z) = X_1 + X_2 \in F_1 + F_2 \Longrightarrow \mathbb{R}^3 \subset F_1 + F_2.$$

We conclude

$$\mathbb{R}^3 = F_1 + F_2.$$

From the direct sum, we know,

$$\dim_{\mathbb{R}}(\mathbb{R}^3) = \dim_{\mathbb{R}}(F_1) + \dim_{\mathbb{R}}(F_2) \Longrightarrow \dim_{\mathbb{R}}(F_2) = 3 - 2 = 1.$$

Exercise No. 2 : Solution

1. (a) **Computing** A^2**,** A^3**and** A^n.

We have

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A^3 = A^2 \cdot A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O_3.$$

Hence, for all integers $n \geq 3$, we have

$$A^n = A^{n-3} \cdot A^3 = A^{n-3} \cdot O_3 = O_3 \text{..................}(*)$$

2. **(a) Explicit Form of** $M(x)$. For all $x \in \mathbb{R}$, we have

$$M(x) = I_3 + xA + \frac{x^2}{2}A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + x \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{x^2}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & \frac{1}{2}x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) (j) Let $x, y \in \mathbb{R}$, then

$$M(x).M(y) = \begin{pmatrix} 1 & x & \frac{1}{2}x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y & \frac{1}{2}y^2 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y & xy + \frac{1}{2}x^2 + \frac{1}{2}y^2 \\ 0 & 1 & x+y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y & \frac{1}{2}(x+y)^2 \\ 0 & 1 & x+y \\ 0 & 0 & 1 \end{pmatrix} = M(x+y).$$

Alternative Method : Let $x, y \in \mathbb{R}$, using the identity $(*)$, we obtain

$$M(x) \cdot M(y) = \left(I_3 + xA + \frac{x^2}{2}A^2 \right) \cdot \left(I_3 + yA + \frac{y^2}{2}A^2 \right)$$

$$= I_3 + yA + \frac{y^2}{2}A^2 + xA + xyA^2 + \frac{x^2}{2}A^2 = I_3 + (x+y)A + \left(xy + \frac{x^2}{2} + \frac{y^2}{2} \right) A^2 =$$

$$= I_3 + (x+y)A + \frac{1}{2}(x+y)^2 A^2 = M(x+y).$$

(c) (j) Let $x \in \mathbb{R}$, then

$$M(x).M(x') = I_3 \Longrightarrow M(x+x') = I_3 = M(0)$$
$$\Longrightarrow M(x+x') = M(0)$$
$$\Longrightarrow x + x' = 0$$
$$\Longrightarrow x' = -x \in \mathbb{R}.$$

(jj) **Inverse of M(x).** Let $x \in \mathbb{R}$. The matrix $M(x)$ is invertible if and only if there exists a matrix $B \in \mathcal{M}_3(\mathbb{R})$ such that

$$M(x) \cdot B = B \cdot M(x) = I_3.$$

Moreover, if $M(x)$ is invertible, then B is unique and $B = M(x)^{-1}$. We have

$$I_3 = M(0) = M(x-x) = M(x) \cdot M(-x)$$

Therefore, the matrix $M(x)$ is invertible and its inverse is

$$M(x)^{-1} = M(-x) = \begin{pmatrix} 1 & -x & \frac{1}{2}x^2 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}.$$

(d) The determinant of $M(x)$ is the product of the diagonal entries, then $\det(M(x)) = 1 \cdot 1 \cdot 1 = 1$.

Exercise No. 3 : Solution :

1. (a) **Kernel of f :** Let $P = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]$. Then,

$$P' = a_1 + 2a_2X, P'' = 2a_2, f(P) = a_1 - 2a_2$$

We obtain

$$\ker(f) = \left\{P = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X] : f(P) = 0_{\mathbb{R}_2[X]}\right\} = \{P \in \mathbb{R}_2[X] : a_1 = 2a_2\} = \left\{P = a_0 + a_2(2X + X^2) : a_0, a_2\right.$$

$$= span(L_1 = \left\{R_1 = 1, R_2 = 2X + X^2\right\})$$

Moreover, it is easy to verify that the family L_1 is linearly independent, thus L_1 forms a basis of $\ker(f)$ and $\dim(\ker(f)) = 2$.

(b) **Image of f.** we have

$$Im(f) = \left\{f(P) = a_1 - 2a_2 : P = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]\right\} = \{f(P) = (a_1 - 2a_2) \cdot 1 : a_1, a_2 \in \mathbb{R}\}$$

$$= span(L_2 = \{T_1 = 1\})$$

Since, $T_1 = 1 \neq 0_{\mathbb{R}_2[X]}$, so L_2 is linearly independent, and thus L_2 forms a basis of $Im(f)$ and $\dim(Im(f)) = 1$.

2. The matrix associated with the linear mapping f, denoted by $A = \underset{B}{Mat}(f)$, is the matrix of size $m \times n$ where ($m = \dim(F = \mathbb{R}_2[X] = 3), n = \dim(E = \mathbb{R}_2[X]) = 3$) and whose columns are the coordinates of the vectors $f(P_0), f(P_1)$ and $f(P_2)$ expressed in the basis B. That is

$$A = \underset{B}{Mat}(f) = \underset{B}{Mat}\begin{pmatrix} f(P_0) & f(P_1) & f(P_2) \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}).$$

Where

$$\begin{aligned} f(P_0) &= f(1) = 0 = 0 \cdot P_0 + 0 \cdot P_1 + 0 \cdot P_2 \\ f(P_1) &= f(X) = 1 = 1 \cdot P_0 + 0 \cdot P_1 + 0 \cdot P_2 \\ f(P_2) &= f(X^2) = -2 = -2 \cdot P_0 + 0 \cdot P_1 + 0 \cdot P_2 \end{aligned}$$

we conclude

$$A = \underset{B}{Mat}(f) = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. **(a) Change of basis matrix** : The change-of-basis matrix $H = \underset{B \longrightarrow B'}{P}$ from B to B' (from an old basis B to a new basis B') (also called the transition matrix), is the 3×3 square matrix whose columns are the coordinates of the vectors of the new basis expressed in the old basis B. That is

$$H = \underset{B \longrightarrow B'}{P} = \underset{B}{Mat}(Q_0, Q_1, Q_2) \in \mathcal{M}_3(\mathbb{R})$$

where

$$\begin{aligned} Q_0 &= X - 1 = -1 \cdot P_0 + 1 \cdot P_1 + 0 \cdot P_2 \\ Q_1 &= X^2 - 1 = -1 \cdot P_0 + 0 \cdot P_1 + 1 \cdot P_2 \\ Q_2 &= X^2 + 1 = 1 \cdot P_0 + 0 \cdot P_1 + 1 \cdot P_2 \end{aligned}$$

we get

$$H = \underset{B \longrightarrow B'}{P} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}).$$

(b) (i) **Find the scalars** $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$. We have

$$P = \lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 \Longrightarrow (-\lambda_0 - \lambda_1 + \lambda_2) + \lambda_0 X + (\lambda_1 + \lambda_2) X^2 = a_0 + a_1 X + a_2 X^2$$

$$\Longrightarrow \begin{cases} -\lambda_0 - \lambda_1 + \lambda_2 = a_0 \\ \lambda_0 = a_1 \\ \lambda_1 + \lambda_2 = a_2 \end{cases}$$

$$\Longrightarrow \left[\lambda_0 = a_1, \lambda_1 = \frac{1}{2}(a_2 - a_1 - a_0), \lambda_2 = \frac{1}{2}(a_0 + a_1 + a_2)\right] \text{..............}(*)$$

(ii) **Coordinates of** P_0, P_1, P_2 **in basis** B'.

For $P_0 = 1$ $(a_0 = 1, a_1 = 0, a_2 = 0)$, by $(*)$, we have $\left[\lambda_0 = 0, \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}\right]$

For $P_1 = X$ $(a_0 = 0, a_1 = 1, a_2 = 0)$, by $(*)$, we have $\left[\lambda_0 = 1, \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}\right]$

For $P_2 = X^2$ $(a_0 = 0, a_1 = 0, a_2 = 1)$, by $(*)$, we have $\left[\lambda_0 = 0, \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2}\right]$

(c) (i) **Compute** H^{-1}. The matrix H^{-1} is the change-of-basis matrix from the new basis B' to the old basis B. In other words

$$H^{-1} = \underset{B' \longrightarrow B}{P} = \underset{B'}{Mat}(P_0, P_1, P_2) = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(ii) The matrix associated with f relative to the new basis B' is given by

$$A' = \underset{B'}{Mat}(f) = H^{-1} \cdot A \cdot H = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 1 \\ \frac{1}{2} & -1 & -1 \end{pmatrix}$$