3.6 Exercise Solutions

Solution 3.1.

1. ***Associativity:***

   To check if $*$ *is associative, we verify if* $(x * y) * z = x * (y * z)$ *for all* $x, y, z \in \mathbb{R}$.

   Let's calculate:

   $$x * y = x + y - xy,$$

   $$(x * y) * z = (x + y - xy) * z = x + y + z - xy - xz - yz + xyz.$$

   Now calculate:

   $$y * z = y + z - yz,$$

   $$x * (y * z) = x * (y + z - yz) = x + (y + z - yz) - x(y + z - yz).$$

   Simplifying gives:

   $$x + y + z - xy - xz - yz + xyz,$$

   which equals $(x * y) * z$. *Hence,* $*$ *is associative.*

2. ***Commutativity:***

   To check if $*$ *is commutative, we verify if* $x * y = y * x$ *for all* $x, y \in \mathbb{R}$.

   Calculating $y * x$ *gives:*

   $$y * x = y + x - yx = x + y - xy = x * y.$$

   Therefore, $*$ *is commutative.*

3. ***Neutral Element:***

   We seek a real number e *such that* $x * e = e * x = x$ *for all* $x \in \mathbb{R}$.

   Setting $x * e = x$ *gives:*

   $$x + e - xe = x.$$

   Simplifying gives $e - xe = 0$, *so* $e(1 - x) = 0$. *For all* $x \neq 1$, *we have* $e = 0$.

   Therefore, the neutral element e *is* 0.

4. ***Inverse Element:***
We look for a real number y *such that* $x*y = y*x = e$*, where* e *is the neutral element.*

Solving $x * y = 0$ *gives:*

$$x + y - xy = 0.$$

Rearranging gives:

$$y - xy = -x,$$

$$y(1-x) = -x.$$

So,

$$y = \frac{-x}{1-x}.$$

Hence, the inverse of x *under* $*$ *is* $\frac{-x}{1-x}$*, provided* $x \neq 1$.

5. ***Formula for*** n***-th Power:***
We consider the operation $*$ *defined on* $\mathbb{R}$ *by:*

$$x * y = x + y - xy, \quad \text{for all } x, y \in \mathbb{R}.$$

We aim to derive a formula for x^{*n}*, defined as:*

$$x^{*n} = \underbrace{x * x * \cdots * x}_{n \text{ times}}.$$

Special cases

- *For* $n = 1$*:*
$$x^{*1} = x.$$
- *For* $n = 2$*:*
$$x^{*2} = x * x = x + x - x^2 = 2x - x^2.$$
- *For* $n = 3$*:*
$$x^{*3} = x^{*2} * x = (2x - x^2) * x.$$
By computation, we find:
$$(2x - x^2) * x = (2x - x^2) + x - (2x - x^2)x = 3x - 3x^2 + x^3.$$

General formula

From the special cases, we observe that the n-th power can be written as:

$$x^{*n} = nx - \binom{n}{2}x^2 + \binom{n}{3}x^3 - \cdots + (-1)^{n-1}\binom{n}{n}x^n,$$

where the coefficients are the binomial coefficients.

Proof by induction

- ***Base case:*** *For $n = 1$, we have $x^{*1} = x$, which matches the formula.*
- ***Inductive step:*** *Assume that:*

$$x^{*n} = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}x^k.$$

We want to show that:

$$x^{*(n+1)} = \sum_{k=1}^{n+1}(-1)^{k-1}\binom{n+1}{k}x^k.$$

By definition:

$$x^{*(n+1)} = x^{*n} * x = x^{*n} + x - x^{*n} \cdot x.$$

Substituting the formula for x^{*n} and simplifying, we recover exactly the terms corresponding to:

$$\sum_{k=1}^{n+1}(-1)^{k-1}\binom{n+1}{k}x^k.$$

This completes the proof by induction.

Final formula

The n-th power for the operation $*$ is given by:

$$x^{*n} = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}x^k,$$

where $\binom{n}{k}$ is the binomial coefficient.

Solution 3.2.

We are given the binary operation $*$ on $\mathbb{R}$ defined by

$$a * b = \ln(e^a + e^b).$$

We will now examine the properties of this operation.

1. Commutativity *To check if* $*$ *is commutative, we need to verify if*

$$a * b = b * a \quad \textit{for all} \quad a, b \in \mathbb{R}.$$

Using the definition of the operation:

$$a * b = \ln(e^a + e^b) \quad \textit{and} \quad b * a = \ln(e^b + e^a).$$

Since $e^a + e^b = e^b + e^a$*, we have:*

$$a * b = b * a.$$

Thus, the operation $*$ *is* ***commutative****.*

2. Associativity *To check if* $*$ *is associative, we need to verify if*

$$(a * b) * c = a * (b * c) \quad \textit{for all} \quad a, b, c \in \mathbb{R}.$$

We calculate both sides using the definition of $*$*:*

- Left-hand side:

$$(a * b) * c = \ln(e^a + e^b) * c = \ln\left(e^{\ln(e^a+e^b)} + e^c\right) = \ln\left((e^a + e^b) + e^c\right)$$

- Right-hand side:

$$a * (b * c) = a * \ln(e^b + e^c) = \ln\left(e^a + e^{\ln(e^b+e^c)}\right) = \ln\left(e^a + (e^b + e^c)\right)$$

Since the two expressions are equal, the operation is ***associative****.*

3. Neutral Element *To find a neutral element* $e \in \mathbb{R}$*, we need to solve the equation*

$$a * e = a \quad \textit{for all} \quad a \in \mathbb{R}.$$

Using the definition of $*$*:*

$$a * e = \ln(e^a + e^e) = a.$$

This simplifies to:

$$e^a + e^e = e^a \quad \Rightarrow \quad e^e = 0.$$

Since $e^e = 0$ *has no real solution, there is* ***no neutral element****.*

4. ***Regular Elements*** *An element* x *is regular if there exists an inverse* x^{-1} *such that*

$$x * x^{-1} = e.$$

Since there is no neutral element, there are ***no regular elements****.*

Conclusion *- The operation* $*$ *is* ***commutative****. - The operation* $*$ *is* ***associative****. - There is* ***no neutral element****. - There are* ***no regular elements****.*

Solution 3.3.

Solution *We are given the operation* $\star$ *defined on* $\mathbb{R}^+$ *as:*

$$\forall x, y \in \mathbb{R}^+, \quad x \star y = \sqrt{x^2 + y^2}.$$

We will analyze the properties step by step.

1. Commutativity *To check if the operation* $\star$ *is commutative, compute:*

$$x \star y = \sqrt{x^2 + y^2}, \quad y \star x = \sqrt{y^2 + x^2}.$$

Since addition is commutative, $x^2 + y^2 = y^2 + x^2$*, it follows that:*

$$x \star y = y \star x.$$

Thus, the operation $\star$ *is* ***commutative****.*

2. ***Associativity*** *To verify associativity, we need to check if:*

$$(x \star y) \star z = x \star (y \star z).$$

Compute $(x \star y) \star z$***:***

$$x \star y = \sqrt{x^2 + y^2}, \quad (x \star y) \star z = \sqrt{(\sqrt{x^2 + y^2})^2 + z^2} = \sqrt{x^2 + y^2 + z^2}.$$

Compute $x \star (y \star z)$***:***

$$y \star z = \sqrt{y^2 + z^2}, \quad x \star (y \star z) = \sqrt{x^2 + (\sqrt{y^2 + z^2})^2} = \sqrt{x^2 + y^2 + z^2}.$$

Since both expressions are equal:

$$(x \star y) \star z = x \star (y \star z).$$

*Thus, the operation $\star$ is **associative**.*

3. ***Existence of a Neutral Element*** *A neutral element $e \in \mathbb{R}^+$ satisfies:*

$$x \star e = x, \quad \forall x \in \mathbb{R}^+.$$

Using the definition of $\star$:

$$x \star e = \sqrt{x^2 + e^2} = x.$$

Squaring both sides:

$$x^2 + e^2 = x^2 \quad \Longrightarrow \quad e^2 = 0 \quad \Longrightarrow \quad e = 0.$$

The neutral element is therefore $e = 0$.

4. ***Existence of Inverse Elements*** *Let us suppose that a symmetric element exists. However, $x \star x' = \sqrt{x^2 + x'^2} = 0$ then $x^2 + x'^2 = 0$ then then $x = x' = 0$. So if $x > 0$, there is no symmetric element.*

Conclusion

- *The operation $\star$ is **commutative**.*
- *The operation $\star$ is **associative**.*
- *The neutral element is therefore $e = 0$.*
- *If $x > 0$, there is no symmetric element.*

Solution 3.4.

1. ***Proving that*** $(\mathbb{R}^* \times \mathbb{R}, \star)$ ***is a Group: Closure:*** *Given two elements (x, y) and (x', y') in $\mathbb{R}^* \times \mathbb{R}$, the operation $\star$ is defined as:*

$$(x, y) \star (x', y') = (xx', xy' + y).$$

Since $x, x' \in \mathbb{R}^*$ and $y, y' \in \mathbb{R}$, it follows that $xx' \in \mathbb{R}^*$ and $xy' + y \in \mathbb{R}$, so the result is in $\mathbb{R}^* \times \mathbb{R}$. Therefore, the operation is closed.

Associativity: *We need to prove that:*

$$((x, y) \star (x', y')) \star (x'', y'') = (x, y) \star ((x', y') \star (x'', y'')).$$

First, compute the left-hand side:

$$(x, y) \star (x', y') = (xx', xy' + y),$$

and then:

$$(xx', xy' + y) \star (x'', y'') = (xx'x'', (xy' + y)y'' + (xy' + y)).$$

Now, compute the right-hand side:

$$(x', y') \star (x'', y'') = (x'x'', x'y'' + y'),$$

and then:

$$(x, y) \star (x'x'', x'y'' + y') = (x(x'x''), x(x'y'' + y') + y).$$

Since both sides are equal, the operation $\star$ is associative.

Neutral Element: *We look for a neutral element (e_1, e_2) such that:*

$$(x, y) \star (e_1, e_2) = (x, y) \quad \text{and} \quad (e_1, e_2) \star (x, y) = (x, y).$$

From the equation:

$$(x, y) \star (e_1, e_2) = (xe_1, xe_2 + y),$$

we must have $e_1 = 1$ and $e_2 = 0$.

Thus, the neutral element is $(1, 0)$.

Inverses: *We look for the inverse of (x, y) such that:*

$$(x, y) \star (x', y') = (1, 0).$$

From the equation:

$$(xx', xy' + y) = (1, 0),$$

we get:

$$xx' = 1 \quad \text{and} \quad xy' + y = 0.$$

Thus, $x' = \frac{1}{x}$ and $y' = -\frac{y}{x}$.

Therefore, the inverse of (x, y) *is* $\left(\frac{1}{x}, -\frac{y}{x}\right)$.

2. Is the operation $\star$ ***Commutative?***

To check if the operation is commutative, we need to verify if:

$$(x, y) \star (x', y') = (x', y') \star (x, y).$$

We have:

$$(x, y) \star (x', y') = (xx', xy' + y),$$

and

$$(x', y') \star (x, y) = (x'x, x'y + y').$$

For the operation to be commutative, we must have:

$$xx' = x'x \quad \text{and} \quad xy' + y = x'y + y'.$$

The first condition $xx' = x'x$ *holds because multiplication in* $\mathbb{R}^*$ *is commutative. However, the second condition* $xy' + y = x'y + y'$ *is not always true, as the terms involve different variables. Thus, the operation is* ***not commutative****.*

3. Simplifying $(x, y)^n$***:***

To compute $(x, y)^n$ *using the operation* $\star$*, we observe the following pattern:*

$$(x, y)^n = (x^n, nxy + y(1 + 2 + \cdots + (n - 1))).$$

The sum $1 + 2 + \cdots + (n - 1)$ *is the sum of the first* $n - 1$ *integers, which is given by:*

$$\frac{(n - 1)n}{2}.$$

Thus, we have:

$$(x, y)^n = (x^n, nxy + \frac{(n - 1)n}{2} y).$$

Summary:

- $(\mathbb{R}^* \times \mathbb{R}, \star)$ *is a group.*
- *The operation is* ***not commutative****.*
- $(x, y)^n = (x^n, nxy + \frac{(n-1)n}{2} y)$.

Solution 3.5.

We define on $\mathbb{R}$, *the composition law* $\circ$ *by:*

$$x \circ y = x + y - 2, \quad \forall x, y \in \mathbb{R}.$$

1. *Show that* $(\mathbb{R}, \circ)$ *is an abelian group.*

   To show that $(\mathbb{R}, \circ)$ *is an abelian group, we need to verify the following properties:*

   - ***Closure:*** *For all* $x, y \in \mathbb{R}$, *we need to show that* $x \circ y \in \mathbb{R}$.

     $$x \circ y = x + y - 2,$$

     which is clearly in $\mathbb{R}$, *so the operation is closed.*

   - ***Associativity:*** *We need to show that for all* $x, y, z \in \mathbb{R}$,

     $$(x \circ y) \circ z = x \circ (y \circ z).$$

     We compute both sides:

     - Left-hand side:

     $$(x \circ y) \circ z = (x + y - 2) \circ z = (x + y - 2) + z - 2 = x + y + z - 4.$$

     - Right-hand side:

     $$x \circ (y \circ z) = x \circ (y + z - 2) = x + (y + z - 2) - 2 = x + y + z - 4.$$

     Since both sides are equal, the operation is associative.

   - ***Neutral element:*** *We need to find the neutral element* $e \in \mathbb{R}$ *such that for all* $x \in \mathbb{R}$,

     $$x \circ e = x \quad \text{and} \quad e \circ x = x.$$

     Using the operation:

     $$x \circ e = x + e - 2 = x \quad \Rightarrow \quad e = 2.$$

     Therefore, the neutral element is $e = 2$.

- ***Inverses:*** *We need to find the inverse of each* $x \in \mathbb{R}$*, i.e., an element* $y \in \mathbb{R}$ *such that:*

$$x \circ y = 2.$$

Using the operation:

$$x \circ y = x + y - 2 = 2 \quad \Rightarrow \quad y = 4 - x.$$

Therefore, the inverse of x *is* $4 - x$*.*

- ***Commutativity:*** *We need to show that* $x \circ y = y \circ x$*. Since*

$$x \circ y = x + y - 2 \quad \text{and} \quad y \circ x = y + x - 2,$$

and since addition is commutative, $x + y = y + x$*, so* $x \circ y = y \circ x$*.*

Since we have shown closure, associativity, the existence of an identity element, inverses, and commutativity, we conclude that $(\mathbb{R}, \circ)$ *is an abelian group.*

2. Let $n \in \mathbb{N}$*. We define* $x(1) = x$ *and* $x(n+1) = x(n) \circ x$*.*

(a) ***Calculate*** $x(2), x(3), x(4)$***:***

We compute $x(2), x(3), x(4)$ *based on the recursive definition of* $x(n)$*:*
- $x(2) = x(1) \circ x = x \circ x = x + x - 2 = 2x - 2$. - $x(3) = x(2) \circ x = (2x-2) \circ x = (2x-2) + x - 2 = 3x - 4$. - $x(4) = x(3) \circ x = (3x-4) \circ x = (3x-4) + x - 2 = 4x - 6$.

Thus, we have:

$$x(2) = 2x - 2, \quad x(3) = 3x - 4, \quad x(4) = 4x - 6.$$

(b) ***Show that for all*** $n \in \mathbb{N}$***,*** $x(n) = nx - 2(n-1)$***:***

We will prove this by induction on n*.*

- ***Base case*** *(*$n = 1$*):*

$$x(1) = x = 1x - 2(1-1) = x.$$

So the base case holds.

- ***Inductive step:*** *Assume the formula holds for some* n*, i.e., assume:*

$$x(n) = nx - 2(n-1).$$

We need to show that:

$$x(n+1) = (n+1)x - 2n.$$

From the definition of $x(n+1)$*:*

$$x(n+1) = x(n)\circ x = (nx-2(n-1))\circ x = (nx-2(n-1))+x-2 = (n+1)x-2n.$$

Thus, the formula holds for $n+1$*, completing the induction.*

Therefore, for all $n \in \mathbb{N}$*, we have:*

$$x(n) = nx - 2(n-1).$$

3. *Let* $A = \{x \in \mathbb{R} \mid x \text{ is even}\}$*. Show that* $(A, \circ)$ *is a subgroup of* $(\mathbb{R}, \circ)$*.*

   We need to verify that $(A, \circ)$ *is a subgroup of* $(\mathbb{R}, \circ)$*. To do this, we must check the following properties:*

   - ***Closure:*** *Let* $x, y \in A$*. Since* x *and* y *are even, we have* $x = 2m$ *and* $y = 2n$ *for some* $m, n \in \mathbb{Z}$*. Now, check if* $x \circ y \in A$*:*

     $$x \circ y = x + y - 2 = 2m + 2n - 2 = 2(m + n - 1).$$

     Since $m + n - 1$ *is an integer,* $x \circ y$ *is even. Thus,* A *is closed under* $\circ$*.*
   - ***Identity element:*** *The identity element of* $(\mathbb{R}, \circ)$ *is 2. We check if* $2 \in A$*. Since 2 is even, the identity element belongs to* A*.*
   - ***Inverses:*** *Let* $x \in A$*. Since* x *is even,* $x = 2m$ *for some* $m \in \mathbb{Z}$*. The inverse of* x *in* $(\mathbb{R}, \circ)$ *is* $4 - x$*. We need to check if the inverse is also in* A*:*

     $$4 - x = 4 - 2m = 2(2 - m),$$

     which is even. Therefore, the inverse of x *is also in* A*.*
   - ***Commutativity:*** *Since* $(\mathbb{R}, \circ)$ *is commutative,* $(A, \circ)$ *inherits commutativity.*

   Since $(A, \circ)$ *is closed, contains the identity element, has inverses for all its elements, and is commutative, it is a subgroup of* $(\mathbb{R}, \circ)$*.*

Solution 3.6.

1.

2. ***To show that $Z(G)$ is a subgroup of G, we need to verify the following properties:***

   - ***Closure:*** *Let $x, z \in Z(G)$. We need to show that $x \cdot z \in Z(G)$, i.e., $(x \cdot z) \cdot y = y \cdot (x \cdot z)$ for all $y \in G$.*
     Since $x \in Z(G)$ and $z \in Z(G)$, we have:

     $$x \cdot y = y \cdot x \quad \text{and} \quad z \cdot y = y \cdot z \quad \text{for all } y \in G.$$

     Now, for $x \cdot z$, we compute:

     $$(x \cdot z) \cdot y = x \cdot (z \cdot y) = x \cdot (y \cdot z) = (x \cdot y) \cdot z = (y \cdot x) \cdot z = y \cdot (x \cdot z),$$

     which shows that $x \cdot z \in Z(G)$. Thus, $Z(G)$ is closed under the group operation.

   - ***Identity element:*** *The identity element $e \in G$ satisfies $e \cdot y = y \cdot e = y$ for all $y \in G$. Since the identity element commutes with every element of G, we have $e \in Z(G)$.*

   - ***Inverses:*** *Let $x \in Z(G)$. We need to show that the inverse of x, denoted x^{-1}, is also in $Z(G)$. Since $x \in Z(G)$, we know that $x \cdot y = y \cdot x$ for all $y \in G$.*
     To show that $x^{-1} \in Z(G)$, we compute:

     $$x^{-1} \cdot y = (x \cdot x^{-1}) \cdot y = x \cdot (x^{-1} \cdot y) = (x \cdot y) \cdot x^{-1} = y \cdot x^{-1} \quad \text{for all } y \in G.$$

     Thus, $x^{-1} \in Z(G)$, and every element of $Z(G)$ has an inverse in $Z(G)$.

   Therefore, $Z(G)$ is a subgroup of G.

3. ***Show that G is commutative if and only if $Z(G) = G$.***

   - ***If G is commutative, then $Z(G) = G$:***
     If G is commutative, then for all $x, y \in G$, we have $x \cdot y = y \cdot x$. Thus, every element of G commutes with every other element of G, which means that $Z(G) = G$.

   - ***If $Z(G) = G$, then G is commutative:***

If $Z(G) = G$, then every element $x \in G$ commutes with every element $y \in G$, i.e., $x \cdot y = y \cdot x$ for all $x, y \in G$. This means that G is commutative.

Therefore, G is commutative if and only if $Z(G) = G$.

Solution 3.7.

1. ***Show that f_a is an endomorphism of the group $(G, \cdot)$.***

   To show that f_a is an endomorphism of G, we need to check that f_a preserves the group operation. That is, we need to verify that for all $x, y \in G$,

   $$f_a(x \cdot y) = f_a(x) \cdot f_a(y).$$

   Compute both sides:

   $$f_a(x \cdot y) = a \cdot (x \cdot y) \cdot a^{-1}.$$

   On the other hand:

   $$f_a(x) \cdot f_a(y) = (a \cdot x \cdot a^{-1}) \cdot (a \cdot y \cdot a^{-1}).$$

   Using the associativity of the group operation:

   $$(a \cdot x \cdot a^{-1}) \cdot (a \cdot y \cdot a^{-1}) = a \cdot x \cdot (a^{-1} \cdot a) \cdot y \cdot a^{-1} = a \cdot x \cdot y \cdot a^{-1}.$$

   Hence, we have:

   $$f_a(x \cdot y) = f_a(x) \cdot f_a(y).$$

   Therefore, f_a is an endomorphism of G.

2. ***Verify that for all $a, b \in G$, $f_a \circ f_b = f_{a \cdot b}$.***

   We want to show that $f_a \circ f_b = f_{a \cdot b}$, i.e., for all $x \in G$,

   $$f_a(f_b(x)) = f_{a \cdot b}(x).$$

   First, compute $f_a(f_b(x))$:

   $$f_b(x) = b \cdot x \cdot b^{-1},$$

so

$$f_a(f_b(x)) = a \cdot (b \cdot x \cdot b^{-1}) \cdot a^{-1} = (a \cdot b) \cdot x \cdot (a \cdot b)^{-1}.$$

Now, compute $f_{a \cdot b}(x)$*:*

$$f_{a \cdot b}(x) = (a \cdot b) \cdot x \cdot (a \cdot b)^{-1}.$$

Therefore, we have:

$$f_a(f_b(x)) = f_{a \cdot b}(x).$$

Thus, $f_a \circ f_b = f_{a \cdot b}$.

3. ***Show that*** f_a ***is bijective and determine its inverse function.***

   To show that f_a *is bijective, we need to prove that it is both injective and surjective.*

   - ***Injectivity:*** *To show that* f_a *is injective, we need to prove that if* $f_a(x) = f_a(y)$*, then* $x = y$*. Suppose:*

     $$f_a(x) = f_a(y) \quad \Rightarrow \quad a \cdot x \cdot a^{-1} = a \cdot y \cdot a^{-1}.$$

     Multiply both sides by a^{-1} *on the left and* a *on the right:*

     $$a^{-1} \cdot (a \cdot x \cdot a^{-1}) \cdot a = a^{-1} \cdot (a \cdot y \cdot a^{-1}) \cdot a \quad \Rightarrow \quad x = y.$$

     Hence, f_a *is injective.*

   - ***Surjectivity:*** *To show that* f_a *is surjective, we need to prove that for every* $z \in G$*, there exists an* $x \in G$ *such that* $f_a(x) = z$*. We want to find* x *such that:*

     $$a \cdot x \cdot a^{-1} = z.$$

     Multiply both sides by a^{-1} *on the left and* a *on the right:*

     $$a^{-1} \cdot (a \cdot x \cdot a^{-1}) \cdot a = a^{-1} \cdot z \cdot a \quad \Rightarrow \quad x = a^{-1} \cdot z \cdot a.$$

     Therefore, for any $z \in G$*, we can find* $x = a^{-1} \cdot z \cdot a$ *such that* $f_a(x) = z$*. Hence,* f_a *is surjective.*

   Since f_a *is both injective and surjective, it is bijective.*

   To find the inverse of f_a*, we need to find a function* $f_{a^{-1}}$ *such that:*

   $$f_{a^{-1}}(f_a(x)) = x \quad \text{and} \quad f_a(f_{a^{-1}}(x)) = x.$$

We compute:

$$f_{a^{-1}}(f_a(x)) = f_{a^{-1}}(a \cdot x \cdot a^{-1}) = a^{-1} \cdot (a \cdot x \cdot a^{-1}) \cdot a = x.$$

Therefore, the inverse of f_a is $f_{a^{-1}}$, and we have:

$$f_a^{-1} = f_{a^{-1}}.$$

Solution 3.8.

1. ***Show that $(A, \star, \diamond)$ is a ring.*** *To show that $(A, \star, \diamond)$ is a ring, we need to verify the following properties:*

   - $\star$ *is associative.*
   - $\diamond$ *is associative.*
   - $\star$ *is commutative.*
   - $\diamond$ *is distributive over $\star$.*
   - 0 *is the identity element for $\star$.*
   - 1 *is the identity element for $\diamond$.*

   1. $\star$ is associative: *We need to show that $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in A$.*

   $$(a \star b) \star c = (a + b + 1) \star c = (a + b + 1) + c + 1 = a + b + c + 2.$$

   $$a \star (b \star c) = a \star (b + c + 1) = a + (b + c + 1) + 1 = a + b + c + 2.$$

   Since both sides are equal, $\star$ is associative.

   2. $\diamond$ is associative: *We need to show that $(a \diamond b) \diamond c = a \diamond (b \diamond c)$ for all $a, b, c \in A$.*

   $$(a \diamond b) \diamond c = (a \cdot b + a + b) \diamond c = (a \cdot b + a + b) \cdot c + (a \cdot b + a + b) + c.$$

   Simplifying this:

   $$= a \cdot b \cdot c + a \cdot c + b \cdot c + a + b + c.$$

Now compute $a \diamond (b \diamond c)$*:*

$$a \diamond (b \diamond c) = a \diamond (b \cdot c + b + c) = a \cdot (b \cdot c + b + c) + a + (b \cdot c + b + c).$$

Simplifying this:

$$= a \cdot b \cdot c + a \cdot b + a \cdot c + a + b \cdot c + b + c.$$

Re-arranging terms:

$$= a \cdot b \cdot c + a \cdot c + b \cdot c + a + b + c.$$

Hence, both sides are equal, so $\diamond$ *is associative.*

3. $\star$ ***is commutative:*** *We need to show that* $a \star b = b \star a$ *for all* $a, b \in A$*.*

$$a \star b = a + b + 1 \quad \text{and} \quad b \star a = b + a + 1.$$

Since addition is commutative in A*, we have* $a \star b = b \star a$*, so* $\star$ *is commutative.*

4. ***Distributivity of*** $\diamond$ ***over*** $\star$***:*** *We need to show that* $a \diamond (b \star c) = (a \diamond b) \star (a \diamond c)$ *for all* $a, b, c \in A$*.*

$$a \diamond (b \star c) = a \diamond (b + c + 1) = a \cdot (b + c + 1) + a + (b + c + 1).$$

Simplifying:

$$= a \cdot b + a \cdot c + a + a + b + c + 1 = a \cdot b + a \cdot c + 2a + b + c + 1.$$

Now, compute $(a \diamond b) \star (a \diamond c)$*:*

$$(a \diamond b) = a \cdot b + a + b, \quad (a \diamond c) = a \cdot c + a + c.$$

$$(a \diamond b) \star (a \diamond c) = (a \cdot b + a + b) \star (a \cdot c + a + c) = (a \cdot b + a + b) + (a \cdot c + a + c) + 1.$$

Simplifying:

$$= a \cdot b + a \cdot c + a + a + b + c + 1 = a \cdot b + a \cdot c + 2a + b + c + 1.$$

Hence, $a \diamond (b \star c) = (a \diamond b) \star (a \diamond c)$*, so* $\diamond$ *is distributive over* $\star$*.*

5. Identity element for $\star$: *We need to show that* 0 *is the identity element for* $\star$.

$$a \star 0 = a + 0 + 1 = a + 1 \quad \textit{and} \quad 0 \star a = 0 + a + 1 = a + 1.$$

So, 0 *is the identity element for* $\star$.

6. Identity element for $\diamond$: *We need to show that* 1 *is the identity element for* $\diamond$.

$$a \diamond 1 = a \cdot 1 + a + 1 = a + a + 1 = 2a + 1 \quad \textit{and} \quad 1 \diamond a = 1 \cdot a + 1 + a = a + 1 + a = 2a + 1.$$

Thus, 1 *is the identity element for* $\diamond$.

Therefore, $(A, \star, \diamond)$ *is a ring.*

2. ***Show that the map $f : (A, +, \cdot) \to (A, \star, \diamond)$ given by $f(a) = a - 1$ is an isomorphism of rings.***

Solution: *We need to show that* f *is a ring isomorphism. This requires that:*

- f *is a homomorphism, i.e., it preserves both addition and multiplication.*
- f *is bijective.*

1. Homomorphism for $\star$: *We need to show that* $f(a \star b) = f(a) \star f(b)$. *We compute:*

$$f(a \star b) = f(a + b + 1) = (a + b + 1) - 1 = a + b,$$

and

$$f(a) \star f(b) = (a - 1) \star (b - 1) = (a - 1) + (b - 1) + 1 = a + b - 1.$$

Since both sides are equal, f *preserves* $\star$.

2. Homomorphism for $\diamond$: *We need to show that* $f(a \diamond b) = f(a) \diamond f(b)$. *We compute:*

$$f(a \diamond b) = f(a \cdot b + a + b) = a \cdot b + a + b - 1,$$

and

$$f(a) \diamond f(b) = (a - 1) \diamond (b - 1) = (a - 1) \cdot (b - 1) + (a - 1) + (b - 1).$$

Expanding this:

$$= a \cdot b - a - b + 1 + a - 1 + b - 1 = a \cdot b + a + b - 1.$$

Hence, f preserves $\diamond$.

3. ***Bijectivity:*** *The map $f(a) = a - 1$ is clearly bijective because it is a linear map with an inverse $f^{-1}(a) = a + 1$.*

Therefore, f is an isomorphism of rings.

Solution 3.9.
Let $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$.

1. *Show that $(\mathbb{Z}[\sqrt{2}], +, \times)$ is a ring.*
2. *Let $N(a + b\sqrt{2}) = a^2 - 2b^2$.* ***Show that for all $x, y \in \mathbb{Z}[\sqrt{2}]$, we have $N(xy) = N(x)N(y)$.***
3. *Deduce that the invertible elements of $\mathbb{Z}[\sqrt{2}]$ are those of the form $a + b\sqrt{2}$ with $a^2 - 2b^2 = \pm 1$.*

Solution 3.10.

1. *It is sufficient to prove that $\mathbb{Z}[\sqrt{2}]$ is a subring of $(\mathbb{R}, +, \times)$. But $\mathbb{Z}[\sqrt{2}]$ is stable under the addition operation:*

$$(a + b\sqrt{2}) + (a' + b'\sqrt{2}) = (a + a') + (b + b')\sqrt{2}.$$

It is also stable under multiplication:

$$(a + b\sqrt{2}) \times (a' + b'\sqrt{2}) = (aa' + 2bb') + (ab' + a'b)\sqrt{2}.$$

It is stable under negation:

$$-(a + b\sqrt{2}) = -a + (-b)\sqrt{2}.$$

Moreover, $1 \in \mathbb{Z}[\sqrt{2}]$, which completes the proof that $\mathbb{Z}[\sqrt{2}]$ is a subring of $\mathbb{R}$.

2. *Let* $x = a + b\sqrt{2}$ *and* $y = a' + b'\sqrt{2}$*. Using the formula for the product obtained in the previous question, we have:*

$$N(xy) = (aa' + 2bb')^2 - 2(ab' + a'b)^2 = (aa')^2 - 2(ab')^2 - 2(a'b)^2 + 4(bb')^2.$$

On the other hand,

$$N(x) \times N(y) = (a^2 - 2b^2)(a'^2 - 2b'^2) = (aa')^2 - 2(ab')^2 - 2(a'b)^2 + 4(bb')^2.$$

3. *Now, suppose* $x = a + b\sqrt{2}$ *is invertible with inverse* y*. Then,* $N(xy) = N(1) = 1$*, and therefore* $N(x)N(y) = 1$*. Since* $N(x)$ *and* $N(y)$ *are both integers, we must have* $N(x) = \pm 1$*. Conversely, if* $N(x) = \pm 1$*, then using the conjugate:*

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \pm(a - b\sqrt{2}),$$

which shows that $a + b\sqrt{2}$ *is invertible with inverse* $\pm(a - b\sqrt{2})$*.*

Solution 3.11.

We will begin by proving that $\mathbb{Q}(i)$ *is a subring of* $\mathbb{C}$*. To do this, we observe that:*

$$1 \in \mathbb{Q}(i)$$

If $z = a + bi$ *and* $z' = a + bi' \in \mathbb{Q}(i)$*, then:*

$$z - z' = (a - a') + i(b - b') \in \mathbb{Q}(i)$$

and

$$zz' = (a + bi)(a' + bi') = (aa' - bb') + i(ab' + a'b) \in \mathbb{Q}(i).$$

Next, let $z = a + bi \in \mathbb{Q}(i)$ *with* $z \neq 0$*. Then:*

$$\frac{1}{z} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2} \in \mathbb{Q}(i).$$

Thus, $\frac{1}{z}$ *is in* $\mathbb{Q}(i)$*, and therefore every non-zero element of* $\mathbb{Q}(i)$ *has an inverse in* $\mathbb{Q}(i)$*. This completes the proof that* $\mathbb{Q}(i)$ *is a field.*