3.5 Exercises

Exercice 3.1.
Consider on $\mathbb{R}$ the binary operation $*$ defined by $x*y = x+y-xy$. Is this operation associative, commutative? Does it have a neutral element? Does a real number x have an inverse under this operation? Provide a formula for the n-th power of an element x under this operation.

Exercice 3.2.
We define an internal composition law $*$ on $\mathbb{R}$ by

$$\forall (a,b) \in \mathbb{R}^2, \quad a*b = \ln(e^a + e^b).$$

What are its properties? Does it have a neutral element? Are there regular elements?

Exercice 3.3.
We are given an internal composition law defined by:

$$\forall x, y \in \mathbb{R}^+, x \star y = \sqrt{x^2 + y^2}$$

Show that this operation is commutative, associative, and that there exists a neutral element. Show that no element has an inverse for this operation.

Exercice 3.4.
We define, for (x,y) and (x',y') in $\mathbb{R}^* \times \mathbb{R}$, the operation $\star$ by:

$$(x,y) \star (x',y') = (xx', xy' + y).$$

Prove that $(\mathbb{R}^* \times \mathbb{R}, \star)$ is a group. Is it commutative? Simplify $(x,y)^n$ for all $(x,y) \in \mathbb{R}^* \times \mathbb{R}$ and for all $n \in \mathbb{N}^*$.

Exercice 3.5.
We define on $\mathbb{R}$, the composition law $\circ$ by:

$$x \circ y = x + y - 2, \quad \forall x, y \in \mathbb{R}.$$

1. *Show that $(\mathbb{R}, \circ)$ is an abelian group.*
2. *Let $n \in \mathbb{N}$. We define $x(1) = x$ and $x(n+1) = x(n) \circ x$.*
    (a) *Calculate $x(2), x(3)$, and $x(4)$.*

   (b) *Show that for all* $n \in \mathbb{N}$: $x(n) = nx - 2(n-1)$.

3. *Let* $A = \{x \in \mathbb{R} \mid x \text{ is even}\}$. *Show that* $(A, \circ)$ *is a subgroup of* $(\mathbb{R}, \circ)$.

Exercice 3.6.

Exercise 3: Let $(G, \cdot)$ *be a group, and denote by* $Z(G) = \{x \in G \mid \forall y \in G, xy = yx\}$ *the center of* G.

1. *Show that* $Z(G)$ *is a subgroup of* G.

2. *Show that* G *is commutative if and only if* $Z(G) = G$.

Exercice 3.7.

Let $(G, \cdot)$ *be a non-commutative group with neutral element* e. *For* $a \in G$, *define a function* $f_a : G \to G$ *by*

$$f_a(x) = a \cdot x \cdot a^{-1}$$

1. *Show that* f_a *is an endomorphism of the group* $(G, \cdot)$.

2. *Verify that for all* $a, b \in G$, $f_a \circ f_b = f_{a \cdot b}$.

3. *Show that* f_a *is bijective and determine its inverse function.*

Exercice 3.8.

Let $(A, +, \cdot)$ *be a ring with identities* 0 *and* 1 *for addition and multiplication respectively. We define the following operations* $\star$ *and* $\diamond$ *on* A:

$$\forall a, b \in A, \quad a \star b = a + b + 1$$

$$\forall a, b \in A, \quad a \diamond b = a \cdot b + a + b$$

1. *Show that* $(A, \star, \diamond)$ *is a ring.*

2. *Show that the map* $f : (A, +, \cdot) \to (A, \star, \diamond)$ *given by* $f(a) = a - 1$ *is an isomorphism of rings.*

Exercice 3.9.

Show that $\mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}$ *is a field.*