2.4 Exercices

Exercice 2.1.

Consider the following sets: $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$ *and*

$C = \{2, 4, 6, 8, 10\}$. *Find the following sets:* $(A \cup B)$, $(B \cap C)$, $(A - B)$ *and* $(A \Delta C)$.

Exercice 2.2.

Let $E = \{a,\ b,\ c\}$ *be a set. Can we write:)*
1 $a \in E$, *2)* $a \subset E$, *3)* $\{a\} \subset E$, *4)* $\emptyset \in E$,*5)* $\emptyset \subset E$, *6)* $\{\emptyset\} \subset E$*?*

Exercice 2.3.

Let A, B, *and* C *be three subsets of a set* E. *Prove that:*

1. $A \setminus B = A \cap B^c$.
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
4. $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Exercice 2.4. *(Power set)*

Let $E = \{a, b, c, d\}$. *Find the power set* $\mathcal{P}(E)$ *of* E, *which is the set of all subsets of* E. *Give an example of a partition of* E, *which is a collection of non-empty disjoint subsets of* E *whose union equals* E.

Exercice 2.5.

1. *What are the images of the sets* $\mathbb{R}$, $[0, 2\pi]$, $[0, \frac{\pi}{2}]$, *and the inverse image of the sets* $[0, 1]$, $[3, 4]$, $[1, 2]$ *under the function* $f(x) = \sin(x)$*?*
2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = x^2 + 1$. *Consider the sets* $A = [-3, 2]$ *and* $B = [0, 4]$. *Compare* $f(A \setminus B)$ *and* $f(A) \setminus f(B)$.
3. *What condition must function* f *satisfy so that* $f(A \setminus B) = f(A) \setminus f(B)$*?*

Exercice 2.6.

Determine whether the following relations are reflexive, symmetric, antisymmetric, and transitive:

1. $E = \mathbb{Z}$ *and* $x\mathcal{R}y \Leftrightarrow |x| = |y|$.

2. $E = \mathbb{R} \setminus \{0\}$ *and* $x\mathcal{R}y \Leftrightarrow xy > 0$.

3. $E = \mathbb{Z}$ *and* $x\mathcal{R}y \Leftrightarrow x - y$ *is even.*

Identify among the above examples which relations are orders and which are equivalence relations.

Exercice 2.7.

Determine whether the following relations are reflexive, symmetric, antisymmetric, and transitive:

1. $E = \mathbb{R}$ *and* $x\mathcal{R}y \Leftrightarrow x = -y$.

2. $E = \mathbb{R}$ *and* $x\mathcal{R}y \Leftrightarrow \cos^2(x) + \sin^2(y) = 1$.

3. $E = \mathbb{N}$ *and* $x\mathcal{R}y \Leftrightarrow \exists p, q \geq 1$ *such that* $y = px^q$ *(where* p *and* q *are integers).*

Identify among the above examples which relations are orders and which are equivalence relations.

Exercice 2.8.

Let $E = \mathbb{Z}$*, the set of all integers. Consider the following relations on* E*:*

1. *Relation* $\sim_1$ *defined by* $x \sim_1 y$ *if and only if* $x + y$ *is even.*

2. *Relation* $\sim_2$ *defined by* $x \sim_2 y$ *if and only if* x *and* y *have the same remainder when divided by 5.*

3. *Relation* $\sim_3$ *defined by* $x \sim_3 y$ *if and only if* $x - y$ *is a multiple of 7.*

For each relation $\sim_i$ *(where* $i = 1, 2, 3$*):*

1. *Determine if* $\sim_i$ *is an equivalence relation on* $\mathbb{Z}$*. Explain why or why not.*

2. *If* $\sim_i$ *is an equivalence relation, identify the equivalence classes of* $\mathbb{Z}$ *under* $\sim_i$.

Exercice 2.9.

Let $\mathcal{R}$ *be an equivalence relation on a non-empty set* E*. Show that*

$$\forall x, y \in E, \quad x\mathcal{R}y \quad \Leftrightarrow \quad \dot{x} = \dot{y}.$$

Exercice 2.10.

Let $\mathbb{N}^*$ denote the set of positive integers. Define the relation $\mathcal{R}$ on $\mathbb{N}^*$ by $x\mathcal{R}y$ if and only if x divides y.

1. *Show that $\mathcal{R}$ is a partial order relation on $\mathbb{N}^*$.*
2. *Is $\mathcal{R}$ a total order relation?*
3. *Describe the sets $\{x \in \mathbb{N}^* \mid x\mathcal{R}5\}$ and $\{x \in \mathbb{N}^* \mid 5\mathcal{R}x\}$.*
4. *Does $\mathbb{N}^*$ have a least element? A greatest element?*

Exercice 2.11.

Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = x^2 + x - 2$.

1. *Give the definition of $f^{-1}(\{4\})$. Calculate $f^{-1}(\{4\})$.*
2. *Is the function f bijective?*
3. *Give the definition of $f([-1, 1])$. Calculate $f([-1, 1])$.*
4. *Give the definition of $f^{-1}([-2, 4])$. Calculate $f^{-1}([-2, 4])$.*

Exercice 2.12.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \frac{2x}{1+x^2}$.

1. *Is f injective? Is f surjective?*
2. *Show that $f(\mathbb{R}) = [-1, 1]$.*
3. *Show that the restriction $g : [-1, 1] \longrightarrow [-1, 1]$ defined by $g(x) = f(x)$ is a bijection.*

Exercice 2.13.

Let $f : E \to F$, $g : F \to G$, and $h = g \circ f$.

1. *Show that if h is injective, then f is injective. Also, show that if h is surjective, then g is surjective.*
2. *Show that if h is surjective and g is injective, then f is surjective.*
3. *Show that if h is injective and f is surjective, then g is injective.*

Exercice 2.14.

Let E be a set, and A and B be two subsets of E. Prove the following properties:

1. $\phi_A + \phi_{A^c} = 1$, *where* $A^c = E \setminus A$.

2. $\phi_{A\cap B} = \phi_A \cdot \phi_B$.

3. $\phi_{A\cup B} = \phi_A + \phi_B - \phi_A \cdot \phi_B$.

4. $\phi_{A\setminus B} = \phi_A(1 - \phi_B)$.

where ϕ_A *is the indicator function of* A*, defined as*

$$\phi_A : E \longrightarrow \{0,1\}, \quad x \longmapsto \phi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$