Chapter 2

Sets, relations and functions

2.1 Sets

2.1.1 Definitions

Definition 2.1.1.
A set is a collection of objects known as elements. An element can be almost anything, such as numbers, functions, or lines. A set is a single object that can contain many elements. Since the elements are those that distinguish one set from another, one method that is used to write a set is to list its elements and surround them with braces. This is called the ***roster method*** *of writing a set, and the list is known as a roster. The braces signify that a set has been defined. For example, the set of all integers between 1 and 10 inclusive is*

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Read this as "the set containing 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10." The set of all integers between 1 and 10 exclusive is

$$\{2, 3, 4, 5, 6, 7, 8, 9\}$$

If A is a set and a is an element of A, write $a \in A$. The notation $a, b \in A$ means $a \in A$ and $b \in A$. If c is not an element of A, write $c \notin A$.

Here's another way to define sets : a collection of elements that satisfy a property. We then write : $E = \{x, P(x)\}$, ($E = \{-1 \leq x \leq 1\} = [-1, 1]$)

Definition 2.1.2.

1. *We denote* $\emptyset$ *as the empty set, which contains no elements.*
2. *A set with one element is called a singleton.*
3. *A set with two (distinct) elements is called a pair.*
4. *The cardinality of a set* E*, denoted* $card(E) = |E|$*, is the number (finite or infinite) of elements in* E*.*

Let A and B be sets. These sets are equal if they contain exactly the same elements. This is denoted by $A = B$. This means that if an element is in A, it must also be in B, and conversely, if an element is in B, it must also be in A. Formally,

$$A = B \text{ if and only if } (\forall x,\ x \in A \iff x \in B)$$

Definition 2.1.3. *(Inclusion)*
If A *and* B *are sets, we say that* A *is included in* B *(or* A *is a subset of* B*) if every element of* A *is also an element of* B*. This is denoted as:*

$$A \subseteq B$$

If A *is a subset of* B *and* A *is not equal to* B *(i.e.,* A *contains fewer elements than* B*), then* A *is a proper subset of* B*, denoted as:*

$$A \subsetneq B$$

In other words:

- $A \subseteq B$ *means all elements of* A *are in* B*.*
- $A \subsetneq B$ *means all elements of* A *are in* B *and* A *is not equal to* B*.*

Remark 2.1.1.
We always have

1. $E \subseteq E$ *(reflexivity).*
2. *If* $F \subseteq E$ *and* $G \subseteq F$*, then* $G \subseteq E$ *(transitivity).*

3. $(E = F) \Leftrightarrow [(E \subseteq F)$ *and* $(F \subseteq E)]$ *(antisymmetry)*

Definition 2.1.4. *(Complement)*
The complement of a set F with respect to a universal set E is the set of all elements in E that are not in F. It is denoted by F^c or $\overline{F}$.

$$F^c = \{x \in E,\ x \notin F\}. \tag{2.1}$$

Proposition 2.1.1.
We always have

1. $(F^c)^c = F$.

2. $F \subseteq G \Leftrightarrow G^c \subseteq F^c$

Proof 2.1.1.

1. *Obvious.*

2. *Suppose $F \subseteq G$. Let $x \in G^c$. Then $x \notin G$, which implies $x \notin F$ (because $F \subseteq G$), and thus $x \in F^c$. Therefore, $G^c \subseteq F^c$.*

2.1.2 Operations on sets

Definition 2.1.5. *(Intersection)*
The intersection of two sets E and F is the set $E \cap F$ of elements x that are in both E and F. We say that two sets E and F are disjoint if $E \cap F = \emptyset$.

$$E \cap F = \{x \mid x \in E \text{ and } x \in F\}. \tag{2.2}$$

Proposition 2.1.2.
We always have:

1. $E \cap F = F \cap E$ *(commutativity).*

2. $E \cap (F \cap G) = (E \cap F) \cap G$ *(associativity).*

3. $(E \subset F \cap G) \Leftrightarrow [(E \subset F)$ *and* $(E \subset G)]$

Definition 2.1.6. *(Union)*

The union of two sets E and F is the set $E \cup F$ of elements x that are in E, in F, or in both.

$$E \cup F = \{x \mid x \in E \text{ or } x \in F\}. \tag{2.3}$$

Example 2.1.1.

Consider the following sets:

$$E = \{1, 2, 3, 4, 5\}$$

$$F = \{4, 5, 6, 7, 8\}$$

***Intersection** $E \cap F$:*

$$E \cap F = \{4, 5\}$$

***Union** $E \cup F$:*

$$E \cup F = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

Proposition 2.1.3.

We always have:

1. $(F \cap G)^c = F^c \cup G^c$.
2. $(F \cup G)^c = F^c \cap G^c$.

Proof 2.1.2.

1. *To prove the relation $(F \cap G)^c = F^c \cup G^c$, we will demonstrate the subset inclusions in both directions.*

   ***1. Show that** $(F \cap G)^c \subseteq F^c \cup G^c$:*

   Let $x \in (F \cap G)^c$. By definition of the complement, this means $x \notin F \cap G$.

   - *Since $x \notin F \cap G$, it must be that x is not in F or x is not in G. In set notation, this is:*

     $$x \notin F \text{ or } x \notin G$$

   - *This implies that $x \in F^c$ or $x \in G^c$. Therefore, $x \in F^c \cup G^c$.*

   Thus:

   $$x \in (F \cap G)^c \implies x \in F^c \cup G^c$$

So:

$$(F \cap G)^c \subseteq F^c \cup G^c$$

2. Show that $F^c \cup G^c \subseteq (F \cap G)^c$***:***

Let $x \in F^c \cup G^c$. *This means* $x \in F^c$ *or* $x \in G^c$.

- *If* $x \in F^c$, *then* $x \notin F$.
- *If* $x \in G^c$, *then* $x \notin G$.
- *In either case,* x *is not in both* F *and* G. *Hence,* $x \notin F \cap G$, *which means:*

$$x \in (F \cap G)^c$$

Thus:

$$x \in F^c \cup G^c \implies x \in (F \cap G)^c$$

So:

$$F^c \cup G^c \subseteq (F \cap G)^c$$

Conclusion:

Since we have shown both inclusions:

$$(F \cap G)^c \subseteq F^c \cup G^c \quad \textit{and} \quad F^c \cup G^c \subseteq (F \cap G)^c$$

We conclude:

$$(F \cap G)^c = F^c \cup G^c$$

2. *Similar to (1).*

Definition 2.1.7. *(Difference)*
If E *and* F *are two sets, the difference* $E \backslash F$ *between* E *and* F *is the set of elements in* E *that are not in* F. *The symmetric difference* $E \triangle F$ *of* E *and* F *is given by:*

$$E \triangle F = (E \backslash F) \cup (F \backslash E). \tag{2.4}$$

Example 2.1.2.
Consider the following sets:

$$E = \{1, 2, 3, 4, 5\}$$

$$F = \{4, 5, 6, 7, 8\}$$

We want to find: $E \backslash F$, $F \backslash E$ and $E \triangle F = (E \backslash F) \cup (F \backslash E)$.

Calculations:

$$E \backslash F = \{1, 2, 3\}$$

$$F \backslash E = \{6, 7, 8\}$$

$$E \triangle F = (E \backslash F) \cup (F \backslash E) = \{1, 2, 3, 6, 7, 8\}$$

Definition 2.1.8. *(Partition of a Set)*
A partition $\mathcal{A} = \{A_1, A_2, A_3, \ldots, A_n\}$ of a set E is a collection of subsets of E such that:

1. $\forall i, A_i \neq \emptyset$,
2. $\forall i, j$ *such that* $i \neq j$, $A_i \cap A_j = \emptyset$,
3. $\bigcup_{i=1}^{n} A_i = E$

Example 2.1.3.
Consider the set:

$$E = \{1, 2, 3, 4, 5, 6\}$$

We want to find a partition of this set. One possible partition is:

$$\mathcal{A} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$$

Verification:

To confirm that $\mathcal{A}$ is indeed a partition of E, we check the following:

1. ***Non-empty subsets:*** *Each subset in the partition is non-empty.*
   - $\{1, 2\}$ *is non-empty.*
   - $\{3, 4\}$ *is non-empty.*
   - $\{5, 6\}$ *is non-empty.*
2. ***Disjoint subsets:*** *No two subsets overlap.*
   - $\{1, 2\} \cap \{3, 4\} = \emptyset$
   - $\{1, 2\} \cap \{5, 6\} = \emptyset$
   - $\{3, 4\} \cap \{5, 6\} = \emptyset$

3. ***Union of subsets equals the original set:*** *The union of all subsets in the partition must equal the original set E.*

$$\{1,2\} \cup \{3,4\} \cup \{5,6\} = \{1,2,3,4,5,6\} = E$$

Thus, $\mathcal{A} = \{\{1,2\},\{3,4\},\{5,6\}\}$ is a valid partition of E.

Definition 2.1.9. *(Cartesian Product)*
The Cartesian product of two sets E and F is the set

$$E \times F = \{(x,y) \mid x \in E \text{ and } y \in F\}. \tag{2.5}$$

The diagonal of a set E is

$$\triangle = \{(x,x) \mid x \in E\} \subset E \times E. \tag{2.6}$$

Example 2.1.4. *Consider the following sets:*

$$E = \{1,2\}$$

$$F = \{a,b\}$$

The Cartesian product $E \times F$ is given by:

$$E \times F = \{(x,y) \mid x \in E \text{ and } y \in F\}$$

To find $E \times F$, we create all possible ordered pairs where the first element comes from E and the second element comes from F: Therefore:

$$E \times F = \{(1,a),(1,b),(2,a),(2,b)\}$$

2.2 Relations

2.2.1 Definitions

Definition 2.2.1. *(Relation)*
A relation from a set A to a set B is any correspondence $\mathcal{R}$ that links elements of A to elements of B in some way.

1. *We say that* A *is the domain and* B *is the codomain of the relation* $\mathcal{R}$.
2. *If* x *is related to* y *by the relation* $\mathcal{R}$*, we say that* x *is related to* y *by* $\mathcal{R}$*; or* (x, y) *satisfies the relation* $\mathcal{R}$*, and we write:* $x\mathcal{R}y$ *or* $\mathcal{R}(x, y)$. *Otherwise, we write:* $x\not\mathcal{R}y$ *or* $\not\mathcal{R}(x, y)$.
3. *A relation from* A *to* A *is called a relation on* A.

Example 2.2.1.

1. *Let* E *be the set of teachers at Mila University, and* F *be the set of students at Mila University. We define a relation* $\mathcal{R}$ *from* E *to* F *by stating that*

$$\forall (x, y) \in E \times F,\ x\mathcal{R}y \Leftrightarrow x \text{ is the teacher of } y. \tag{2.7}$$

2. *Other examples of human relations include: "is a brother of," "has the same age as."*
3. *Let* $A = B = \mathbb{Z}$. *We define a relation* $\mathcal{R}$ *on* $\mathbb{Z}$ *by stating that*

$$\forall (x, y) \in \mathbb{Z}^2,\ x\mathcal{R}y \Leftrightarrow 2 \mid (x - y). \tag{2.8}$$

   Thus, $1\mathcal{R}7$ *since* 2 *divides* $-6 = (1 - 7)$. *Note that* $18\not\mathcal{R}7$ *since* 2 *does not divide* $11 = (18 - 7)$.
4. *The correspondence* $\mathcal{R}'$ *that links digits to vowels used to write the digit in full text is a relation from the set* $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *to the set* $\{a, e, i, o, u, y\}$. *For example,* $0\mathcal{R}'e$, $0\mathcal{R}'o$, $0\not\mathcal{R}'a$, $9\not\mathcal{R}'y$, $6\mathcal{R}'i$, *and* $1\not\mathcal{R}'u$.

Definition 2.2.2. *(Graph of a Relation)*
Let $\mathcal{R}$ *be a relation from a set* A *to a set* B. *The graph of* $\mathcal{R}$ *(denoted* $G_{\mathcal{R}}$*) is the set defined by:*

$$G_{\mathcal{R}} = \{(x, y) \in A \times B \mid x\mathcal{R}y\} \tag{2.9}$$

Example 2.2.2.

1. *Referring back to the relation* $\mathcal{R}$ *from the previous example, we have:* $(1, 7) \in G_{\mathcal{R}}$ *and* $(18, 7) \notin G_{\mathcal{R}}$.

2. *For the relation $\mathcal{R}'$ given in the previous example, we have:*

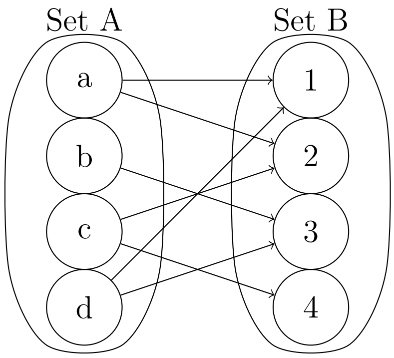
$$G_{\mathcal{R}'} = \{(0,e),(0,o),(1,o),(1,e),(2,o),(3,e),(4,o),(4,u),$$

$$(5,i),(5,e),(6,i),(7,e),(8,e),(8,i),(9,i),(9,e)\}.$$

Remark 2.2.1.
Given two relations $\mathcal{R} = (A, B, G_{\mathcal{R}})$ and $\mathcal{R}' = (A', B', G_{\mathcal{R}'})$, the statement "the relations $\mathcal{R}$ and $\mathcal{R}'$ are equal" means that $A = A'$, $B = B'$, and $G_{\mathcal{R}} = G_{\mathcal{R}'}$: same domain, same codomain, and same graph.

Definition 2.2.3. *(Inverse of relation)*
The inverse of a relation $\mathcal{R}$ from A to B, denoted $\mathcal{R}^{-1}$, is a relation from B to A where $y\mathcal{R}^{-1}x$ if and only if $x\mathcal{R}y$.

Definition 2.2.4. *(A sagittal diagram)*
A sagittal diagram is a graphical representation used to visualize the relationships between elements in different sets. In the context of relations, it can illustrate how elements from one set relate to elements in another set through a given relation.



Sagittal Diagram of a Relation

2.2.2 Properties of a Binary Relation on a Set

We now focus on relations where the domain is the same as the codomain.

Let A be a set and $\mathcal{R}$ is binary relation on A.

Definition 2.2.5. *(Reflexive Relation)*
A relation $\mathcal{R}$ on a set A is called reflexive if every element is related to itself. Formally:

$$\forall x \in A, \; x\mathcal{R}x \tag{2.10}$$

Definition 2.2.6. *(Symmetric Relation)*
A relation $\mathcal{R}$ on a set A is called symmetric if whenever x is related to y, then y is also related to x. Formally:

$$\forall x, y \in A,\ x\mathcal{R}y \implies y\mathcal{R}x \tag{2.11}$$

Definition 2.2.7. *(Antisymmetric Relation)*
A relation $\mathcal{R}$ on a set A is called antisymmetric if whenever x is related to y and y is related to x, then x must be equal to y. Formally:

$$\forall x, y \in A,\ (x\mathcal{R}y \text{ and } y\mathcal{R}x) \implies x = y \tag{2.12}$$

Definition 2.2.8. *(Transitive Relation)*
A relation $\mathcal{R}$ on a set A is called transitive if whenever x is related to y and y is related to z, then x is also related to z. Formally:

$$\forall x, y, z \in A,\ (x\mathcal{R}y \text{ and } y\mathcal{R}z) \implies x\mathcal{R}z \tag{2.13}$$

Example 2.2.3.

1. *The relation $=$ on the set of integers $\mathbb{Z}$ is reflexive, symmetric, antisymmetric, and transitive.*
2. *The relation $\leq$ on the set of integers $\mathbb{Z}$ is reflexive, antisymmetric, and transitive, but not symmetric.*
3. *The relation $<$ on the set of integers $\mathbb{Z}$ is transitive and antisymmetric, but not reflexive or symmetric.*

Example 2.2.4.

1. *Let $A = B = \mathbb{Z}$ and $G_{\mathcal{R}} = \{(a, b) \in \mathbb{Z}^2 : 2 \mid (a - b)\}$. Then, $\mathcal{R}$ is reflexive, symmetric, transitive, but not antisymmetric.*
2. *Given the universe $\mathcal{U}$, the relation of inclusion, which relates two subsets of $\mathcal{U}$ (i.e., $X \subseteq Y$), is also reflexive, transitive, and antisymmetric, but not symmetric.*
3. *Let the relation $\mathcal{R}$ be defined on $\mathbb{Z}$ by: $x\mathcal{R}y \Leftrightarrow x$ divides y.*

(a) *Let $x \in \mathbb{Z}$. We have x divides x (even 0 divides 0). Thus, for all $x \in \mathbb{Z}$, $x\mathcal{R}x$, so $\mathcal{R}$ is reflexive.*

(b) *Let $x, y \in \mathbb{Z}$. We have $x\mathcal{R}y \Rightarrow$ (x divides y) $\nRightarrow$ (y divides x). For example, 1 divides 4 and 4 does not divide 1, so $\mathcal{R}$ is not symmetric.*

(c) *Let $x, y \in \mathbb{Z}$. We have $(x\mathcal{R}y) \wedge (y\mathcal{R}x) \Rightarrow$ ((x divides y) $\wedge$ (y divides x)) $\nRightarrow$ $(x = y)$. For example, 1 divides -1 and -1 divides 1, but $1 \neq -1$. Thus, $\mathcal{R}$ is not antisymmetric.*

(d) *Let $x, y, z \in \mathbb{Z}$. We have $(x\mathcal{R}y) \wedge (y\mathcal{R}z) \Rightarrow$ ((x divides y) $\wedge$ (y divides z)) $\Rightarrow$ (x divides z) $\Rightarrow x\mathcal{R}z$. Hence, $\mathcal{R}$ is transitive.*

2.2.3 Equivalence Relation

Definition 2.2.9.
Let $\mathcal{R}$ be a relation on a set A.

1. *$\mathcal{R}$ is called an* ***equivalence relation*** *if $\mathcal{R}$ is reflexive, symmetric, and transitive.*

2. *If $\mathcal{R}$ is an equivalence relation, then*

   (a) *For each $a \in A$, the set $\dot{a} = \{x \in A \mid x\mathcal{R}a\}$ is called the* ***equivalence class*** *of a modulo $\mathcal{R}$.*

   (b) *The set $A/\mathcal{R} = \{\dot{a} \mid a \in A\}$ is called the* ***quotient set*** *of A by $\mathcal{R}$.*

Example 2.2.5.
Consider the set $A = \{1, 2, 3, 4, 5, 6\}$ and the relation $\mathcal{R}$ defined by $a\mathcal{R}b$ if and only if $2/(a - b)$.

To verify that $\mathcal{R}$ is an equivalence relation:

- ***Reflexivity:*** *For any $a \in A$, $2/(a - a)$.*
- ***Symmetry:*** *If $a\mathcal{R}b$, then $2/(a - b)$, hence $2/(b - a)$.*
- ***Transitivity:*** *If $a\mathcal{R}b$ and $b\mathcal{R}c$, then $2/(a-b)$ and $2/(b-c)$, hence $2/(a-c)$.*

Therefore, $\mathcal{R}$ is indeed an equivalence relation on A.

Equivalence Classes:

- $\dot{1} = \{1, 3, 5\}$
- $\dot{2} = \{2, 4, 6\}$

Quotient Set:

$$A/\mathcal{R} = \{\dot{1}, \dot{2}\} = \{\{1, 3, 5\}, \{2, 4, 6\}\}$$

This example illustrates how the set A is partitioned into equivalence classes under the relation $\mathcal{R}$.

2.2.4 Order Relation

Definition 2.2.10.
Let $\mathcal{R}$ be a relation on a set A.

1. *$\mathcal{R}$ is called an **order relation** if $\mathcal{R}$ is reflexive, antisymmetric and transitive.*
2. (a) *If $\mathcal{R}$ is an order relation, we often write $\leq_{\mathcal{R}}$ instead of $\mathcal{R}$.*

   (b) *$\leq_{\mathcal{R}}$ is called a **total order relation** if*

   $$\forall x, y \in A : (x \leq_{\mathcal{R}} y) \vee (y \leq_{\mathcal{R}} x)$$

   .

   (c) *$\leq_{\mathcal{R}}$ is called a **partial order relation** if*

   $$\exists x, y \in A : \ ((x \not\leq_{\mathcal{R}} y) \wedge (y \not\leq_{\mathcal{R}} x))$$

   .

Remark 2.2.2.
Two elements x and y are said to be comparable by $\leq_{\mathcal{R}}$, if $x \leq_{\mathcal{R}} y$ or $y \leq_{\mathcal{R}} x$.

Example 2.2.6.
Consider the set $A = \{1, 2, 3, 4, 5\}$ and define the relation $\leq$ on A such that for any $x, y \in A$, $x \leq y$ if and only if x divides y (i.e., y is divisible by x).

To verify that $\leq$ is a partial order relation:

- ***Reflexivity:*** *For any $x \in A$, $x \leq x$ because any number divides itself.*

- ***Antisymmetry:*** *If $x \leq y$ and $y \leq x$, then both x divides y and y divides x. This implies $x = y$, hence $\leq$ is antisymmetric.*
- ***Transitivity:*** *If $x \leq y$ and $y \leq z$, then x divides y and y divides z, so x divides z. Thus, $x \leq z$, and $\leq$ is transitive.*

Therefore, $\leq$ is indeed an order relation on the set A.

Comparability:

- *Elements 2 and 4 are comparable because $2 \leq 4$ (since 2 divides 4).*
- *Elements 3 and 5 are not comparable because neither 3 divides 5 nor 5 divides 3.*

So $\leq$ is indeed a partial order.

Definition 2.2.11. *(Special Elements)*
Let $\mathcal{R}$ be an order relation on a set E, and let A be a subset of E.

1. *An element $m \in E$ is called a minimum of A if*
   (a) *$m \in A$,*
   (b) *for all $x \in A$, we have $m \leq_{\mathcal{R}} x$. (We also say that m is the smallest element of A.)*
2. *An element $M \in E$ is called a maximum of A if*
   (a) *$M \in A$,*
   (b) *for all $x \in A$, we have $x \leq_{\mathcal{R}} M$. (We also say that M is the largest element of A.)*
3. *An extremum is an element that is either a minimum or a maximum.*
4. *An element $u \in E$ is called a lower bound of A if for all $x \in A$, we have $u \leq_{\mathcal{R}} x$. (We also say that A is bounded below by u.)*
5. *An element $U \in E$ is called an upper bound of A if for all $x \in A$, we have $x \leq_{\mathcal{R}} U$. (We also say that A is bounded above by U.)*
6. *The set A is said to be bounded below in E if A has a lower bound in E; A is said to be bounded above in E if A has an upper bound in E; and A is said to be bounded in E if A is both bounded below and bounded above.*

7. *An element* $v \in E$ *is called a greatest lower bound of* A *(or infimum of* A*) if*
   (a) v *is a lower bound of* A*,*
   (b) *for every lower bound* v' *of* A*, we have* $v' \leq_{\mathcal{R}} v$*. (We denote this as* $v = \inf(A)$*.)*
8. *An element* $V \in E$ *is called a least upper bound of* A *(or supremum of* A*) if*
   (a) V *is an upper bound of* A*,*
   (b) *for every upper bound* V' *of* A*, we have* $V \leq_{\mathcal{R}} V'$*. (We denote this as* $V = \sup(A)$*.)*

Example 2.2.7.
Consider the set $E = \{1, 2, 3, 4, 5\}$ *with the order relation* $\leq$*.*

1. *For subset* $A = \{2, 3, 4\}$*:*
   - 2 *is a minimum of* A *because* $2 \in A$ *and* $2 \leq 2$*,* $2 \leq 3$*,* $2 \leq 4$*.*
   - 4 *is a maximum of* A *because* $4 \in A$ *and* $3 \leq 4$*,* $2 \leq 4$*.*
2. *For subset* $A = \{2, 3\}$*:*
   - 1 *is a lower bound of* A *because* $1 \leq 2$*,* $1 \leq 3$*.*
   - 5 *is an upper bound of* A *because* $2 \leq 5$*,* $3 \leq 5$*.*
   - $\inf(A) = 2$ *(greatest lower bound of* A*).*
   - $\sup(A) = 3$ *(least upper bound of* A*).*

2.3 Functions and mappings.

2.3.1 Functions.

Definition 2.3.1.

- *A relation* f *from* E *to* F *is called a function if every* $x \in E$ *has at most one image* y *in* F*. We also say that* f *is a function and write* $y = f(x)$ *instead of* xfy*. We also write*

$$f : E \to F$$
$$x \mapsto f(x).$$

- *The domain of definition of a function f (denoted D_f) is the set of elements x in E for which $f(x)$ exists.*

Definition 2.3.2.
Let $f : E \to F$ be a function, where A is a subset of E and B is a subset of F.

1. *The image of A under f is defined as*

$$f(A) = \{f(x) \mid x \in A\}.$$

2. *The preimage of B under f is defined as*

$$f^{-1}(B) = \{x \in E \mid f(x) \in B\}.$$

Definition 2.3.3.
The composition of the function $f : E \to F$ and the function $g : F \to G$ is the function

$$g \circ f : E \to G$$

$$x \mapsto g(f(x)).$$

Example 2.3.1.
Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = 2x + 1$$

$$g(x) = x^2$$

To find the composition $g \circ f$, we compute:

$$(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2$$

Therefore, the composition $g \circ f$ is:

$$(g \circ f)(x) = (2x + 1)^2$$

Definition 2.3.4. *(Restriction of a Function)*
Let $f : E \to F$ be a function. The restriction of f to a subset $A \subseteq E$, denoted $f|_A$, is a new function defined as follows:

- *The domain of $f|_A$ is A, i.e., $\text{dom}(f|_A) = A$.*

- *For each $x \in A$, $f|_A(x) = f(x)$.*

In other words, $f|_A$ is the function that maps each element $x \in A$ to the same value $f(x)$ that f maps x to in the original function f.

Example 2.3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. The restriction of f to the interval $[0, 1]$ is denoted $f|_{[0,1]}$ and is defined by:*

$$f|_{[0,1]}(x) = x^2 \quad \text{for } x \in [0, 1].$$

This restriction $f|_{[0,1]}$ is a function from $[0, 1]$ to $\mathbb{R}$, and it retains the same mapping as f but only for the subset $[0, 1]$ of its original domain.

Example 2.3.3. *(Function extension)*
Consider the function $f : \mathbb{R}^* \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$.
• ***Extension of the Domain***
Let's extend the domain of f to $\mathbb{R}$. Define g as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^*, \\ 0 & \text{if } x = 0. \end{cases}$$

Here, g extends f by including $x = 0$, which was not in the original domain of f.

2.3.2 Mapping

Definition 2.3.5.
A function f is a mapping if every element of E maps to exactly one image in F. We denote $\mathcal{F}(E, F)$ the set of all mappings from E to F. A function f is a mapping if and only if its domain of definition is the entire set E.

Proposition 2.3.1.
Let $f : E \to F$ be a mapping.

1. ***For Sets in** E**:***

1. *If $A \subseteq B$, then $f(A) \subseteq f(B)$.*
2. *We always have $f(A \cup B) = f(A) \cup f(B)$.*
3. *We always have $f(A \cap B) \subseteq f(A) \cap f(B)$.*

2. ***For Sets in*** F***:***

   1. *If* $A \subseteq B$*, then* $f^{-1}(A) \subseteq f^{-1}(B)$*.*
   2. *We always have* $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$*.*
   3. *We always have* $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$*.*

3. ***Additional Properties:***

   1. *If* A *is a subset of* E*, then* $A \subseteq f^{-1}(f(A))$*.*
   2. *If* B *is a subset of* F*, then* $f(f^{-1}(B)) \subseteq B$*.*

2.3.3 Injection, surjection, bijection

Let E, F be two sets and $f : E \to F$ be a mapping .

Definition 2.3.6.

1. f *is injective (One-to-One) if every element of* F *has at most one preimage in* E*. In other words:*

$$\forall x, y \in E, \quad f(x) = f(y) \Rightarrow x = y.$$

2. f *is surjective (Onto) if every element of* F *has at least one preimage in* E*. In other words:*

$$\forall y \in F, \quad \exists x \in E \text{ such that } f(x) = y.$$

3. f *is bijective (One-to-One Correspondence) if it is both injective and surjective (every element of* F *has exactly one preimage in* E*.*

We can make the following remarks:

Remark 2.3.1.

1. *A mapping is injective if and only if its inverse relation is a function.*

2. *A mapping f is surjective if and only if its image is equal to its codomain (target set).*

3. *A mapping is bijective if and only if its inverse relation is a mapping.*

Example 2.3.4.

Consider the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$.

Injective (One-to-One):

To show that f is injective, we need to demonstrate that for any $x_1, x_2 \in \mathbb{R}$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. Let $x_1, x_2 \in \mathbb{R}$. Suppose $f(x_1) = f(x_2)$. Then, $2x_1 = 2x_2$. Dividing both sides by 2 gives us $x_1 = x_2$. Therefore, f is injective.

Surjective (Onto):

To show that f is surjective, we need to show that for every $y \in \mathbb{R}$, there exists at least one $x \in \mathbb{R}$ such that $f(x) = y$.

Let $y \in \mathbb{R}$. We want to find $x \in \mathbb{R}$ such that $f(x) = y$. From the definition of f, we have $f(x) = 2x$. Setting $2x = y$, we get $x = \frac{y}{2}$. Since $\frac{y}{2} \in \mathbb{R}$ for all $y \in \mathbb{R}$, f is surjective.

Bijective (One-to-One Correspondence):

Since f is both injective and surjective, it is bijective. This means that every element in $\mathbb{R}$ has a unique preimage under f, and every element in $\mathbb{R}$ is mapped to by at least one element in $\mathbb{R}$.

Therefore, $f(x) = 2x$ is an example of a bijective function from $\mathbb{R}$ to $\mathbb{R}$.

Proposition 2.3.2.

1. *If $f : E \to F$ and $g : F \to G$ are two injective, surjective, or bijective mappings, then $g \circ f : E \to G$ is also injective, surjective, or bijective respectively.*

2. *A mapping $f : E \to F$ is bijective if and only if there exists a mapping $g : F \to E$ such that $g \circ f = \mathrm{Id}_E$ and $f \circ g = \mathrm{Id}_F$. In this case, $g = f^{-1}$.*

3. *If $f : E \to F$ and $g : F \to G$ are two mappings with $g \circ f : E \to G$ injective, then f is also injective. Similarly, if $g \circ f$ is surjective, then g is surjective as well.*

Proof 2.3.1.

1. (a) *Suppose f and g are injective functions. Let $x_1, x_2 \in E$ such that $g \circ f(x_1) = g \circ f(x_2)$. Then, $g(f(x_1)) = g(f(x_2)) \Rightarrow f(x_1) = f(x_2)$ (since g is injective) $\Rightarrow x_1 = x_2$. Hence, $g \circ f$ is injective.*

   (b) *Suppose f and g are surjective functions. Let $z \in G$. Then, there exists $y \in F$ such that $g(y) = z$ (since g is surjective). And since f is surjective, there exists $x \in E$ such that $f(x) = y$. Therefore, there exists $x \in E$ such that $g(f(x)) = z$, implying $g \circ f$ is surjective.*

   (c) *This is evident from parts (a) and (b).*

2. *See section (2.4)*

3. *See section (2.4)*