1.6 Exercise Solutions

Solution 1.1.

1. *The sentence "Paris is in France or Madrid is in China" is a true proposition.*
2. *The sentence "Open the door" is not a proposition because it is a command.*
3. *The sentence "The moon is a satellite of the Earth" is a true proposition.*
4. *The equation* $x + 5 = 7$ *is not a proposition.*
5. *The inequality* $x + 5 > 9$ *for every real number* x *is a false proposition.*

Solution 1.2.
Determine whether each of the following implications is true or false.

1. *If* 0.5 *is an integer, then* $1 + 0.5 = 3$*. True.*
2. *If* $5 > 2$*, then cats can fly. False.*
3. *If* $3 \times 5 = 15$*, then* $1 + 2 = 3$*. True.*
4. *For any real* $x \in \mathbb{R}$*, if* $x \leq 0$*, then* $(x - 1) < 0$*. True.*

Solution 1.3.
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function. Negate the following propositions:*

1. $\exists x \in \mathbb{R}$ *such that* $f(x) = 0$.
   Negation: $\forall x \in \mathbb{R},\ f(x) \neq 0$.
2. $\exists M > 0, \forall A > 0, \exists x \geq A : f(x) \leq M$.
   Negation: $\forall M > 0,\ \exists A > 0,\ \forall x \geq A : f(x) > M$.
3. $\exists x \in \mathbb{R},\ f(x) > 0$.
   Negation: $\forall x \in \mathbb{R},\ f(x) \leq 0$.
4. $\forall \epsilon > 0,\ \exists \eta > 0,\ \forall (x, y) \in I^2,\ (|x - y| \leq \eta \Rightarrow |f(x) - f(y)| > \epsilon)$.
   Negation: $\exists \epsilon > 0,\ \forall \eta > 0,\ \exists (x, y) \in I^2,\ |x - y| \leq \eta \wedge |f(x) - f(y)| \leq \epsilon$.

Solution 1.4.

Consider the statement "for all integers a and b, if $a + b$ is even, then a and b are even."

1. ***Contrapositive:*** *"For all integers a and b, if a or b is odd, then $a + b$ is odd."*
2. ***Converse:*** *"For all integers a and b, if a and b are even, then $a+b$ is even."*
3. ***Negation:*** *"There exist integers a and b such that $a + b$ is even and a is odd or b is odd."*
4. ***Original Statement:*** *False (Counterexample: $a = 1, b = 1$, $a + b = 2$, but both are odd.)*
5. ***Contrapositive:*** *False (The contrapositive is logically equivalent to the original statement.)*
6. ***Converse:*** *True (The sum of two even numbers is always even.)*
7. ***Negation:*** *True*

Solution 1.5.

Prove that if n is an even integer, then n^2 is also an even integer.

Suppose n is an even integer. By definition, this means $n = 2k$ for some integer k. Now, consider n^2:

$$n^2 = (2k)^2 = 4k^2$$

Since k is an integer, $4k^2$ is clearly divisible by 2 (specifically, $4k^2 = 2(2k^2)$). Therefore, n^2 is even. □

Solution 1.6.

Suppose, for the sake of contradiction, that $\sqrt{2}$ is rational. This means we can express $\sqrt{2}$ as $\frac{p}{q}$, where p and q are integers with no common factors (i.e., $\frac{p}{q}$ is in its simplest form).

Then,

$$\sqrt{2} = \frac{p}{q}$$

Squaring both sides gives:

$$2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}$$

Multiplying through by q^2 gives:

$$2q^2 = p^2$$

This implies that p^2 is even (since $2q^2$ is even). From this, we conclude that p itself must be even (because the square of an odd number is odd, and the square of an even number is even).

Let $p = 2k$ for some integer k. Substituting in $p^2 = (2k)^2 = 4k^2$, we get:

$$2q^2 = 4k^2 \implies q^2 = 2k^2$$

This shows that q^2 is even, hence q must also be even (similar reasoning as for p). Now, both p and q are even. However, this contradicts our initial assumption that $\frac{p}{q}$ is in its simplest form (no common factors). If both p and q are even, then $\frac{p}{q}$ can be further reduced by dividing both numerator and denominator by 2, which contradicts the assumption that $\frac{p}{q}$ is already in its simplest form.

Therefore, our initial assumption that $\sqrt{2}$ is rational must be false. Hence, $\sqrt{2}$ is irrational. □

Solution 1.7.

(1) Prove that if n^2 is an even integer, then n is also an even integer.

We will prove this statement by contrapositive.

Contrapositive: *The contrapositive of "If n^2 is even, then n is even" is: "If n is odd, then n^2 is odd."*

Proof: *Let n be an odd integer. By the definition of odd integers, $n = 2k + 1$ for some integer k.*

Now, calculate n^2:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since $2k^2 + 2k$ is an integer, the expression $2(2k^2 + 2k) + 1$ is odd.

Thus, if n is odd, then n^2 is odd. This proves the contrapositive, and hence, if n^2 is even, then n is even.

(2) Prove that if a and b are integers and ab is odd, then both a and b are odd.

We will prove this statement by contrapositive.

Contrapositive: *The contrapositive of "If ab is odd, then both a and b are odd" is: "If either a or b is even, then ab is even."*

Proof: *Assume that either a or b is even. Without loss of generality, assume a is even. Then,* $a = 2k$ *for some integer* k.

Therefore, the product ab becomes:

$$ab = (2k) \times b = 2(kb).$$

Since $2(kb)$ *is divisible by 2, ab is even.*

Thus, if either a or b is even, then ab is even. This proves the contrapositive, and hence, if ab is odd, then both a and b must be odd.

Solution 1.8.

1. *Prove that for all positive integers* n, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

   Proof: *We will prove the statement by mathematical induction.*

   Base Case: *For* $n = 1$,
   $$1 = 1^2$$
   which is true.

   Inductive Step: *Assume the statement holds for some arbitrary positive integer* k*, i.e.,*
   $$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$
   We need to prove it holds for $k + 1$*:*
   $$1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2.$$
   Adding $2(k + 1) - 1 = 2k + 1$ *to both sides gives:*
   $$k^2 + 2k + 1 = (k + 1)^2.$$
   Hence, the statement holds for $k + 1$.

   Therefore, by mathematical induction, for all positive integers n, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. □

2. *Prove that for all positive integers n, $2^n > n$.*

Proof: *We will prove the statement by mathematical induction.*

Base Case: *For $n = 1$,*

$$2^1 = 2 > 1,$$

which is true.

Inductive Step: *Assume the statement holds for some arbitrary positive integer k, i.e.,*

$$2^k > k.$$

We need to prove it holds for $k + 1$:

$$2^{k+1} = 2 \cdot 2^k > 2 \cdot k = k + k \geq k + 1.$$

Therefore, $2^{k+1} > k + 1$.

Hence, by mathematical induction, for all positive integers n, $2^n > n$. □

Solution 1.9.

We are tasked with proving that for all $x \in \mathbb{R}$, the inequality

$$|x - 1| \leq x^2 - x + 1$$

holds. We'll use a ***proof by disjunction of cases****.*

Case 1: $x - 1 \geq 0$ ***(i.e.,*** $x \geq 1$

If $x - 1 \geq 0$, the absolute value becomes:

$$|x - 1| = x - 1.$$

Substituting $|x - 1| = x - 1$ into the inequality:

$$x - 1 \leq x^2 - x + 1.$$

Rearranging terms:

$$x^2 - 2x + 2 \geq 0.$$

Factoring $x^2 - 2x + 2$ (or completing the square):

$$x^2 - 2x + 2 = (x - 1)^2 + 1.$$

Since $(x-1)^2 \geq 0$ *and* $1 > 0$*, it follows that:*

$$x^2 - 2x + 2 \geq 0.$$

Thus, the inequality holds in this case.

Case 2: $x - 1 < 0$ ***(i.e.,*** $x < 1$

If $x - 1 < 0$*, the absolute value becomes:*

$$|x-1| = -(x-1) = 1 - x.$$

Substituting $|x-1| = 1 - x$ *into the inequality:*

$$1 - x \leq x^2 - x + 1.$$

Rearranging terms:

$$x^2 \geq 0.$$

This inequality is always true because $x^2 \geq 0$ *for all* $x \in \mathbb{R}$*.*

Conclusion:

In both cases, the inequality $|x-1| \leq x^2 - x + 1$ *holds for all* $x \in \mathbb{R}$*.*

Solution 1.10.
Prove that the following statement is false: "Every positive integer is the sum of three squares."
Proof: *We will provide a counterexample to show that not every positive integer can be expressed as the sum of three squares.*

Consider the integer $n = 7$*.*

Now, we will check if 7 *can be expressed as* $a^2 + b^2 + c^2$ *where* a, b, c *are integers.*

Testing possible combinations:

$$1^2 + 1^2 + 1^2 = 1 + 1 + 1 = 3,$$
$$1^2 + 1^2 + 2^2 = 1 + 1 + 4 = 6,$$
$$1^2 + 2^2 + 2^2 = 1 + 4 + 4 = 9.$$

None of these combinations yield 7*.*

Therefore, 7 *cannot be expressed as the sum of three squares.*

Since we have found at least one positive integer (specifically 7) that cannot be written as the sum of three squares, the statement "Every positive integer is the sum of three squares" is false. □