1.5 Exercices

Exercice 1.1.
Which of the following sentences are propositions? What are the truth values of those that are propositions?

1. *Paris is in France or Madrid is in China.*
2. *Open the door.*
3. *The moon is a satellite of the Earth.*
4. $x + 5 = 7$.
5. $x + 5 > 9$ *for every real number* x.

Exercice 1.2.
Determine whether each of the following implications is true or false.

1. *If* 0.5 *is an integer, then* $1 + 0.5 = 3$.
2. *If* $5 > 2$, *then cats can fly.*
3. *If* $3 \times 5 = 15$, *then* $1 + 2 = 3$.
4. *For any real* $x \in \mathbb{R}$, *if* $x \leq 0$, *then* $(x - 1) < 0$.

Exercice 1.3.
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function. Negate the following propositions:*

1. $\exists x \in \mathbb{R}$ *such that* $f(x) = 0$.
2. $\exists M > 0, \forall A > 0, \exists x \geq A : f(x) \leq M$.
3. $\exists x \in \mathbb{R},\ f(x) > 0$.
4. $\forall \epsilon > 0,\ \exists \eta > 0$ *such that* $\forall (x, y) \in I^2,\ |x - y| \leq \eta \Rightarrow |f(x) - f(y)| > \epsilon$.

Exercice 1.4.
Consider the statement "for all integers a *and* b, *if* $a + b$ *is even, then* a *and* b *are even":*

1. *Write the contrapositive of the statement.*

2. *Write the converse of the statement.*

3. *Write the negation of the statement.*

4. *Is the original statement true or false? Prove your answer.*

5. *Is the contrapositive of the original statement true or false? Prove your answer.*

6. *Is the converse of the original statement true or false? Prove your answer.*

7. *Is the negation of the original statement true or false? Prove your answer.*

Exercice 1.5. *(Direct Proof)*
Prove that if n is an even integer, then n^2 is also an even integer.

Exercice 1.6. *(Proof by Contradiction)*
Prove that $\sqrt{2}$ is irrational.

Exercice 1.7. *(Proof by Contrapositive)*

1. *Prove that if n^2 is an even integer, then n is also an even integer.*

2. *Prove that if a and b are integers and ab is odd, then both a and b are odd.*

Exercice 1.8. *(Proof by Mathematical Induction)*

1. *Prove that for all positive integers n, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.*

2. *Prove that for all positive integers n, $2^n > n$.*

Exercice 1.9. *(Proof by Cases)*
Show that for all $x \in \mathbb{R}$, the following inequality holds:

$$|x - 1| \leq x^2 - x + 1.$$

Exercice 1.10. *(Counterexample)*
Prove that the following statement is false: "Every positive integer is the sum of three squares."