Solution of series $N°3$

Reminder 1: Solving First−order differential equations

1.**Separable equations:** These are equations which may be written in the form $y' = f(y)g(t)$. To solve, you separate the variables, then integrate, making sure to include one of the constants of integration:

$$\int \frac{1}{f(y)}\,dy = \int g(t)\,dt + C.$$

2.**Linear equations:** These are equations of this form $y' + p(t)y = q(t)$. Solution Steps:

- Find the Integrating Factor: $I(t) = e^{\int p(t)dt}$.
- Multiply the equation by $I(t)$ to transform the left side into a derivative: $\frac{d}{dt}(I(t)y) = I(t)q(t)$.
- Integrate both sides: $I(t)y = \int I(t)q(t)dt + C$.
- Solve for y: $y = \frac{1}{I(t)}\left(\int I(t)q(t)dt + C\right)$.

3. **Homogeneous equations:** A first-order differential equation is homogeneous if it can be written as: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$. Solution Steps:

- Substitute $v = \frac{y}{x}$ so that $y = vx$.
- Differentiate: $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
- Substitute into the equation: $v + x\frac{dv}{dx} = f(v)$.
- Rearrange: $\frac{dv}{f(v) - v} = \frac{dx}{x}$.

- Integrate both sides and solve for y.

4. **Bernoulli equations:** These are a lot like linear equations, they are equations of this form: $y' + p(t)y = q(t)y^n,\ n \in \mathbb{R} - \{0, 1\}$. To solve it, make the substitution $v = y^{1-n}$, so that $v' = (1-n)y^{-n}y'$ in other words $y^{-n}y' = \frac{1}{1-n}v'$. Multiply the original equation by y^{-n}:

$$y^{-n}y' + p(t)y^{1-n} = q(t)$$

Now make the substitution with v: $\frac{1}{1-n}v' + p(t)v = q(t)$. Multiply everything by $1-n$ and you have a linear equation, which you can solve to find v. Once you have v, then use the equation $y = v^{\frac{1}{1-n}}$ to find y.

5. **Riccati equations:** The general Form of the Riccati equation is: $y' = a(x)y^2 + b(x)y + c(x)$. Solution steps:

If a particular solution y_p is known, use the substitution: $y = y_p + \frac{1}{v}$ to transform it into the linear form: $\frac{dv}{dx} + (b + 2ay_p)v = -a$. Solve using the integrating factor.

<u>Exercise 1:</u>

1. $y' - 2xy = (1-2x)e^x,\ y(0) = 5$. The equation is linear, the homogeneous solution y_h is

$$\begin{aligned} y' - 2xy = 0 \implies y' = 2xy \implies \frac{y'}{y} = 2x \\ \implies \ln|y| = \int 2x\ dx = x^2 + c \\ \implies y_h = Ke^{x^2},\ K = \pm e^c. \end{aligned}$$

The particular solution is

$$y_p = K(x)e^{x^2} \implies y_p' = K'(x)e^{x^2} + 2xK(x)e^{x^2}.$$

So

$$\begin{aligned} K'(x)e^{x^2} + 2xK(x)e^{x^2} - 2xK(x)e^{x^2} &= (1-2x)e^x \\ &\Longrightarrow K'(x)e^{x^2} = (1-2x)e^x \\ &\Longrightarrow K'(x) = (1-2x)e^{x-x^2} \\ &\Longrightarrow K(x) = \int (1-2x)e^{x-x^2}\, dx = e^{x-x^2} \end{aligned}$$

hence $y_p = e^{x-x^2}.e^{x^2} = e^x$. Then the generale solution is

$$y = h_h + y_p = Ke^{x^2} + e^x.$$

For $y(0) = 5$, we get $K + 1 = 5 \Longrightarrow K = 4$. So $\boxed{y = 4e^{x^2} + e^x}$

2. $-2x + yy' = 0,\ y(1) = 1$. We have

$$\begin{aligned} 2x + yy' &= 0 \Longrightarrow y\ dy = 2x\ dx \\ &\Longrightarrow \int y\ dy = 2\int x\ dx \\ &\Longrightarrow \frac{y^2}{2} = x^2 + c \\ &\Longrightarrow y^2 = 2x^2 + K,\ where\ K \in \mathbb{R} \end{aligned}$$

We have $y(1) = 1 \Longrightarrow 1 = 2 + K \Longrightarrow K = -1$, then $y^2 = 2x^2 - 1$ or $\boxed{y = \pm\sqrt{2x^2 - 1}}$

3. $xy' + (1+x)y = 0,\ y(1) = 1$. We have

$$\begin{aligned} xy' + (1+x)y = 0 &\Longrightarrow \frac{dy}{y} = -\frac{x}{1+x}\, dx \\ &\Longrightarrow \int \frac{dy}{y} = -\int \frac{x}{1+x}\, dx = -\int 1 - \frac{1}{1+x}\, dx \\ &\Longrightarrow \ln|y| = -x + \ln|1+x| + c \\ &\Longrightarrow y = \pm e^c(1+x)e^{-x} \\ &\Longrightarrow y = K(1+x)e^{-x},\ where\ K \in \mathbb{R} \end{aligned}$$

We have $y(1) = 1 \Longrightarrow 1 = K2e^{-1} \Longrightarrow K = \frac{1}{2}e$. So

$$\boxed{y = \frac{1}{2}(1+x)e^{1-x}}$$

4. $(4-x^2)yy' = 2(1+y^2)$. We have

$$\begin{aligned}
(4-x^2)yy' = 2(1+y^2) &\Longrightarrow \frac{y\ dy}{1+y^2} = \frac{2\ dx}{4-x^2} \\
&\Longrightarrow \int \frac{y\ dy}{1+y^2} = \int \frac{2\ dx}{4-x^2} = \frac{1}{2}\int \frac{1}{2-x} + \frac{1}{2+x}\ dx \\
&\Longrightarrow \frac{1}{2}\ln(1+y^2) = \frac{1}{2}(-\ln|2-x| + \ln|2+x|) + c \\
&\Longrightarrow \ln(1+y^2) = \ln\left|\frac{2+x}{2-x}\right| + c \\
&\Longrightarrow 1+y^2 = K\left|\frac{2+x}{2-x}\right|,\ K \in \mathbb{R}^*
\end{aligned}$$

So $\boxed{y = \pm\sqrt{K\frac{2+x}{2-x} - 1}}$

5. $y' - 2xy = e^{x^2}\sin x,\ y(0) = 1$. The homogeneous equation is $y' - 2xy = 0 \Longrightarrow \frac{y'}{y} = 2x$, so the homogeneous solution given by $\ln|y| = x^2 + c$. Hence $y_h = Ke^{x^2},\ K = \pm e^c$.

The particular solution is $y_p = K(x)e^{x^2} \Longrightarrow y_p' = K'(x)e^{x^2} + 2xK(x)e^{x^2}$, then:

$$\begin{aligned}
K'(x)e^{x^2} + 2xK(x)e^{x^2} - 2xK(x)e^{x^2} &= e^{x^2}\sin x \\
&\Longrightarrow K'(x)e^{x^2} = e^{x^2}\sin x \\
&\Longrightarrow K'(x) = \sin x \\
&\Longrightarrow K(x) = -\cos x
\end{aligned}$$

therefore, $y_p = -\cos xe^{x^2}$, and so the generale solution is given by $y = y_h + y_p = Ke^{x^2} - \cos xe^{x^2}$. For $x = 0 \Longrightarrow K = 2$, so

$$\boxed{y = 2e^{x^2} - \cos xe^{x^2}}$$

6. $xy' + y = x,\ y(2) = 0$. We have

$$\begin{aligned} xy' + y = x &\Longrightarrow \frac{d}{dx}(xy) = x \\ &\Longrightarrow \int d(xy) = \int x\ dx \\ &\Longrightarrow xy = \frac{x^2}{2} + c \\ &\Longrightarrow y = \frac{1}{2}x + \frac{c}{x}. \end{aligned}$$

We have $y(2) = 0 \Longrightarrow 0 = \frac{1}{2}2 + \frac{c}{2} \Longrightarrow c = -2$. So

$$\boxed{y = \frac{1}{2}x - \frac{2}{x}}$$

7. $y' - 2y = -\frac{2}{1+e^{-2x}},\ y(0) = 2$. The integrating factor is

$$v(x) = e^{\int a(x)\ dx} = e^{\int -2\ dx} = e^{-2x}.$$

Multiplication of Equation (7) by e^{-2x} gives

$$e^{-2x}y' - 2e^{-2x}y = -\frac{2e^{-2x}}{1+e^{-2x}}$$

So

$$\frac{d}{dx}(e^{-2x}y) = -\frac{2e^{-2x}}{1+e^{-2x}}$$

then

$$e^{-2x}y = -\int \frac{2e^{-2x}}{1+e^{-2x}}\ dx = \ln(1+e^{-2x}) + c$$

and so

$$y = e^{2x}\ln(1+e^{-2x}) + ce^{2x}.$$

Since $y(2) = 0$, we have

$$2 = e^0 \ln(1+e^0) + ce^0 \Longrightarrow c = 2 - \ln 2$$

Therefore the solution to the initial-value problem is

$$\boxed{e^{2x}\ln(1+e^{-2x}) + (2-\ln 2)e^{2x}}$$

8. $y' - \left(2x - \dfrac{1}{x}\right) y = 1,\ \ on\ ]0,+\infty[$. Resolution of the homogeneous equation: $y' - \left(2x - \dfrac{1}{x}\right) y = 0$. An antiderivative of $a(x) = 2x - \dfrac{1}{x}$ is $A(x) = x^2 - \ln x$, so the solutions of the homogeneous equation are

$$y_h = K \exp\left(x^2 - \ln x\right) = K\frac{1}{x}e^{x^2}$$

for any real constant K.

To find a particular solution, we will use the method of variation of constants for the equation:

$$y' - \left(2x - \frac{1}{x}\right) y = 1$$

We assume a solution of the form:

$$y_p = K(x)\frac{1}{x}e^{x^2}$$

where $K(X)$ is now a function to be determined. So

$$y_p' = K'(x)\frac{1}{x}e^{x^2} + K(x)\left(-\frac{1}{x^2}+2\right)e^{x^2}$$

hence

$$\begin{aligned} K'(x)\frac{1}{x}e^{x^2} + K(x)\left(-\frac{1}{x^2}+2\right)e^{x^2} - \left(2x-\frac{1}{x}\right)K(x)\frac{1}{x}e^{x^2} &= 1 \\ K'(x)\frac{1}{x}e^{x^2} &= 1 \\ K'(x) &= xe^{-x^2} \end{aligned}$$

Thus, by choosing $K(x) = -\frac{1}{2}e^{-x^2}$, we obtain the particular solution:

$$y_p = -\frac{1}{2}e^{-x^2}\frac{1}{x}e^{x^2} = -\frac{1}{2x}$$

Then, the generale solution is geven by:

$$\boxed{y(x) = -\frac{1}{2x} + K\frac{1}{x}e^{x^2},\ K \in \mathbb{R}}$$

9. $xy' + 3y = x^2y^2$. We have

$$xy' + 3y = x^2y^2 \Longrightarrow y^{-2}y' + \frac{3}{x}y^{-1} = x$$

we put $z = y^{1-n} = y^{-1}$, then $z' = -y^{-2}y'$.

Inserting $z = y^{-1}$ and $z' = -y^{-2}y'$ into the differential equation, we get

$$z' - \frac{3}{x}z = x \cdots\cdots (1)$$

The integrating factor is

$$v(x) = e^{\int a(x)\ dx} = e^{\int -\frac{3}{x}\ dx} = \frac{1}{x^3}.$$

Multiplication of Equation (1) by x^3 gives

$$\frac{1}{x^3}z' - \frac{3}{x^4}z = -\frac{1}{x^2}$$

or

$$\frac{d}{dx}\left(\frac{1}{x^3}z\right) = -\frac{1}{x^2}$$

then

$$\frac{1}{x^3}z = -\int \frac{1}{x^2}\ dx = \frac{1}{x} + k$$

or $z = y^{-1} = x^2 + kx^3$, and so

$$\boxed{y = \frac{1}{x^2 + kx^3}}$$

10. $y' + 2xy = -xy^4$. We have

$$y' + 2xy = -xy^4 \Longrightarrow y^{-4}y' + 2xy^{-3} = -x$$

we put $z = y^{1-n} = y^{-3}$, hen $z' = -3y^{-4}y'$.

Inserting $z = y^{-3}$ and $z' = -3y^{-4}y'$ into the differential equation, we get

$$-\frac{1}{3}z' + 2xz = -x$$

or

$$z' - 6xz = 3x \cdots\cdots (2)$$

The integrating factor is

$$v(x) = e^{\int a(x)\ dx} = e^{\int -6x\ dx} = e^{-3x^2}$$

Multiplication of Equation (2) by e^{-3x^2} gives

$$e^{-3x^2}z' - 6xe^{-3x^2}z = 3xe^{-3x^2}$$

or

$$\frac{d}{dx}\left(e^{-3x^2}z\right) = 3xe^{-3x^2}$$

then

$$e^{-3x^2}z = -\frac{1}{2}e^{-3x^2} + C$$

or $z = y^{-3} = -\frac{1}{2} + Ce^{3x^2} \Longrightarrow y = \frac{1}{-\frac{1}{2} + Ce^{3x^2}}$. And so

$$\boxed{y = \left(\frac{2}{Ke^{3x^2} - 1}\right)^{\frac{1}{3}}, \text{ where } K \in \mathbb{R}}$$

Exercise 2:

1. We have $y = 2x \Longrightarrow y' = 2$

$$\begin{aligned}(E) \quad &\Longleftrightarrow \quad (x^2 - 3x + 2)2 - 4x^2 + 6x^2 = 4x^2 - 6x + 4 \\ &= \quad 4x^2 - 6x + 4 = 4x^2 - 6x + 4.\end{aligned}$$

2. The resolution of the equation (E), we put $y = z + 2x \Longrightarrow y' = z' + 2$. By substituting into (E), we obtain

$$(x^2 - 3x + 2)z'z^{-2} - 1 - xz = 0 \cdots (F)$$

The equation (F) is a Bernoulli equation. Dividing equation (F) by z^2 we obtain:

$$(x^2 - 3x + 2)z' - z^2 - xz = 0$$

is Bernoulli equation $\alpha = 2$, then

$$(x^2 - 3x + 2)z'z^{-2} - 1 - xz^{-1} = 0 \cdots (F_1)$$

Then, we set: $t = z^{-1}$

$$t = z^{-1} \Longrightarrow z = t^{-1} \Longrightarrow z' = -\frac{t'}{t^2} \Longrightarrow z'z^{-2} = -t'$$

By replacing z^{-1} with t and $z'z^{-2}$ accordingly in (F_1), we obtain:

$$(x^2 - 3x + 2)t' + xt = -1 \cdots (F_2)$$

F_2 is a first-order linear nonhomogeneous differential equation.

- The homogeneous solution t_h of (F_2):

$$(x^2 - 3x + 2)t' + xt = 0 \cdots (F_3)$$

So

$$\begin{aligned}
(F_3) \iff & (x^3 - 3x + 2)\frac{dt}{dx} = -xt \iff \frac{dt}{t} = \frac{-x}{x^3 - 3x + 2}\, dx \\
\iff & \ln|t| = \int \frac{-x}{x^3 - 3x + 2}\, dx = \int \frac{-x}{(x-1)(x-2)}\, dx \\
= & \int \frac{dx}{x-1} - 2\int \frac{dx}{x-2} \\
= & \ln|x-1| - 2\ln|x-2| + C = \ln\left(\frac{|x-1|}{(x-2)^2}\right) + C,\ C \in \mathbb{R}
\end{aligned}$$

Hence $t_h = K\frac{x-1}{(x-2)^2},\ K \in \mathbb{R}$.

- The particular solution t_p of (F_2). We use the variation of the constant, we set

$$t_p = K(x)\frac{x-1}{(x-2)^2}$$

We replace t with $K(x)\frac{x-1}{(x-2)^2}$ in equation (F_2), we obtain

$$\begin{aligned}
\frac{(x-1)^2}{x-2}K'(x) = -1 \implies & K'(x) = -\frac{x-2}{(x-1)^2} \\
\implies & K(x) = -\int \frac{x-2}{(x-1)^2}\, dx = -\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} \\
= & -\ln|x-1| + \left(-\frac{1}{x-1}\right) + C
\end{aligned}$$

then $t_p = \left(-\ln|x-1| + \left(-\frac{1}{x-1}\right)\right)\left(\frac{x-1}{(x-2)^2}\right)$.

So, the general solution of (F_2) is

$$t = K\frac{x-1}{(x-2)^2} + t_p,\ K \in \mathbb{R}.$$

The general solution of (F) is

$$z = \frac{1}{K\frac{x-1}{(x-2)^2} + t_p}, \ K \in \mathbb{R}.$$

Hence, the general solution of (E) is

$$\boxed{y = \frac{1}{K\frac{x-1}{(x-2)^2} + t_p} + 2x, \ K \in \mathbb{R}.}$$

Reminder 2: Solving Second−order differential equations

A second-order differential equation has the general form: $a(x)y'' + b(x)y' + c(x)y = f(x)$. Where y'' is the second derivative, y' is the first derivative, and $f(x)$ is a given function. There are two main types of second-order differential equations:

1. Homogeneous Equation $f(x) = 0$:

   Solution method

   - Assume a solution of the form: $y_h = e^{rx}$.
   - Substitute into the equation to obtain the characteristic equation: $ar^2 + br + c = 0$.
   - Cases of solutions:
     - Distinct real roots $(r_1 \neq r_2)$:

       $$y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

     - Repeated root $r_1 = r_2 = r$:

       $$y_h = (C_1 + C_2 x)e^{rx}.$$

     - Complex roots $r = \alpha \pm i\beta$:

$$y_h = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$$

2. Non-Homogeneous Equation ($f(x) \neq 0$):

Solution method

- Find the homogeneous solution y_h by solving $ay'' + by' + cy = 0$.
- Find a particular solution y_p using one of the following methods:
  - Method of Undetermined Coefficients: used when $f(x)$ is a polynomial, exponential, or trigonometric function.
  - Variation of Parameters: A general method that uses the fundamental solutions of the homogeneous equation.
- General Solution: $y = y_h + y_p$.

Exercise 3:

1. $y'' + y' - 6y = 4e^x,\ y(0) = 1,\ y'(0) = -22$. The homogeneous solution y_h:

$$y'' + y' - 6y = 0$$

We set $y_h = e^{rx} \Longrightarrow y_h' = re^{rx}$, and $y'' = r^2 e^{rx}$, so

$$r^2 e^{rx} + re^{rx} - 6e^{rx} = 0 \Longrightarrow (r^2 + r - 6)e^{rx} = 0 \Longrightarrow r^2 + r - 6 = 0.$$

Compute $\triangle = 1 + 24 = 25 > 0$, then

$$r_1 = \frac{-1-5}{2} = -3, \text{ and } r_2 = \frac{-1+5}{2} = 2$$

hence the homogeneous soltion given by $y_h = C_1 e^{-3x} + C_2 e^{2x}$, for all $C_1,\ C_2 \in \mathbb{R}$.

Now, we seek the particular solution in the form $y_p = ae^x$, then

$$y_p' = y_p'' = ae^x$$

therefore

$$ae^x + ae^x - 6ae^x = 4e^x \Longrightarrow -4ae^x = 4e^x \Longrightarrow a = -1$$

so $y_p = -e^x$. Then the generale solution is given by:

$$y = y_h + y_p = C_1e^{-3x} + C_2e^{2x} - e^x, \ \forall C_1, \ C_2 \in \mathbb{R}.$$

For $x = 0$, we have

$$\begin{cases} y(0) = 1 \\ \\ y'(0) = -22 \end{cases} \Longrightarrow \begin{cases} C_1 + C_2 - 1 = 1 \\ \\ -3C_1 + 2C_2 - 1 = -22 \end{cases} \Longrightarrow C_1 = 5, \ and \ C_2 = -3.$$

So $\boxed{y = 5e^{-3x} - 3e^{2x} - e^x}$

2. $y'' - 3y' + 2y = (1 - 2x)e^x$. The associated homogeneous equation is:

$$y'' - 3y' + 2y = 0$$

which has the characteristic equation:

$$r^2 - 3r + 2 = 0$$

It has two real roots: $r_1 = 1$ and $r_2 = 2$. Thus, the solutions of the homogeneous equation are:

$$y_h = C_1e^x + C_2e^{2x}, \ C_1, \ C_2 \in \mathbb{R}.$$

And $y_p = x(ax+b)e^x$ because $\alpha = 1$ is a solution of the characteristic equation.

$$\begin{aligned} y_p = (ax^2 + bx)e^x \quad &\Longrightarrow \quad y_p' = \left[ax^2 + (2a + b)x + b\right] e^x \\ &\Longrightarrow \quad y_p'' = \left[ax^2 + (4a + b)x + 2a + 2b\right] e^x \end{aligned}$$

Substituting into equation (2), we obtain:

$$[ax^2 + (4a+b)x + 2a + 2b]\, e^x - 3\,[ax^2 + (2a+b)x + b]\, e^x + 2x(ax+b)e^x =$$

$$(1-2x)e^x$$

$$\iff -2ax + 2a - b = 1 - 2x$$

Thus

$$\Longrightarrow \begin{cases} -2a = -2 \\ 2a - b = 1 \end{cases} \Longrightarrow \begin{cases} a = 1 \\ b = 1 \end{cases}$$

Hence $y_p = (x^2 + x)e^x$. Thus, the solutions of equation (2) are:

$$\boxed{y = y_h + y_p = C_1 e^x + C_2 e^{2x} + (x^2+x)e^x,\ \forall C_1,\ C_2 \in \mathbb{R}}$$

3. $y'' - 2y' + 2y = 5\cos x,\ y(0) = 1,\ y'\left(\frac{\pi}{2}\right) = -2\left(e^{\frac{\pi}{2}} + 1\right)$. The associated homogeneous equation is: $y'' - 2y' + 2y = 0$, which has the characteristic equation: $r^2 - 2r + 2 = 0$. It has two complex roots:

$$r_1 = 1 - i,\ r_2 = 1 + i$$

Thus, the solutions of the homogeneous equation are:

$$y_h = e^x(C_1 \cos x + C_2 \sin x),\ C_1,\ C_2 \in \mathbb{R}.$$

The particular solution in the form $y_p = a_1 \cos x + a_2 \sin x$

$$y_p = a_1 \cos x + a_2 \sin x \implies y_p' = -a_1 \sin x + a_2 \cos x$$

$$\implies y_p'' = -a_1 \cos x - a_2 \sin x$$

Substituting into equation (3), we obtain:

$$-a_1\cos x - a_2\sin x - 2(-a_1\sin x + a_2\cos x) + 2(a_1\cos x + a_2\sin x) = 5\cos x$$

$$\iff (a_1 - 2a_2)\cos x + (2a_1 + a_2)\sin x = 5\cos x$$

so $a_1 = 1$, and $a_2 = -2$, hence

$$y_p = \cos x - 2\sin x.$$

Thus, the generale solution is

$$y = y_h + y_p = e^x(C_1\cos x + C_2\sin x) + \cos x - 2\sin x,\ C_1,\ C_2 \in \mathbb{R}$$

And

$$\begin{cases} y(0) = 2 \\ y'\left(\frac{\pi}{2}\right) = -2\left(e^{\frac{\pi}{2}} - 1\right) \end{cases} \Longrightarrow \begin{cases} C_1 = 1 \\ C_2 = -2 \end{cases}$$

So

$$\boxed{y = e^x(\cos x - 2\sin x) + \cos x - 2\sin x}$$

4. $y'' + 4y = 2\sin x\cos x \iff y'' + 4y = \sin(2x)$. The associated homogeneous equation is:

$$y'' + 4y = 0,$$

which has the characteristic equation:

$$r^2 + 4r = 0.$$

It has two complex conjugate roots:

$$r_1 = 2i, \text{ and } r_1 = -2i$$

Thus, the solutions of the homogeneous equation are:

$$y_h = A\cos(2x) + B\sin(2x),\ \ A,\ \ B \in \mathbb{R}.$$

And since $\alpha + i\beta = 2i$ is a root of the characteristic equation, we seek a particular solution in the form:

$$y_p = x(a\cos(2x) + b\sin(2x))$$

where a and b are constants to be determined. Substituting the expression of y_p into equation (4), we obtain:

$$\begin{gathered} x\left[-4a\cos(2x) - 4b\sin(2x)\right] + 4b\cos(2x) - 4a\sin(2x) + \\ 4x\left[a\cos(2x) + b\sin(2x)\right] = \sin(2x) \\ \Longleftrightarrow 4b\cos(2x) - 4a\sin(2x) = \sin(2x) \end{gathered}$$

From this, we derive the system:

$$\begin{cases} 4b = 0 \\ -4a = 1 \end{cases} \Longleftrightarrow \begin{cases} a = -\dfrac{1}{4} \\ b = 0 \end{cases}$$

And consequently:

$$y_p = -\frac{1}{4}\cos(2x).$$

The general solution of equation (4) is:

$$\boxed{y = A\cos(2x) + B\sin(2x) - \frac{1}{4}\cos(2x),\ \ A,\ \ B \in \mathbb{R}}$$

5. $y'' - 4y' + 4y = xe^{2x},\ y(0) = 1,\ y'(0) = 4$. The associated homogeneous equation is:

$$y'' - 4y' + 4y = 0,$$

which has the characteristic equation:

$$r^2 - 4r + 4 = 0.$$

It has a double root: $r = 2$. Thus, the solutions of the homogeneous equation are:

$$y_h = (C_1 + C_2 x)e^{2x}, \ C_1, \ C_2 \in \mathbb{R}.$$

Since $\alpha = 2$ is a double root of the characteristic equation, we seek a particular solution in the form:

$$y_p = x^2(ax + b)e^{2x}$$

Substituting the expression of y_p into equation (5), we obtain: $a = \frac{1}{6}$, $b = 0$, so $y_p = \frac{1}{6}x^3 e^{2x}$. Hence the generale solution is:

$$\boxed{y = (C_1 + C_2 x)e^{2x} + \frac{1}{6}x^3 e^{2x}, \ C_1, \ C_2 \in \mathbb{R}}$$

6. $y'' + 4y = e^{3x}\cos(2x)$. The associated homogeneous equation is:

$$y'' + 4y = 0$$

which has the characteristic equation:

$$r^2 + 4 = 0$$

Solving for r: $r^2 = -4 \Longrightarrow r = \pm 2i$. Since the characteristic roots are complex conjugates $2i$ and $-2i$, the general solution of the homogeneous equation is given by:

$$y_h = A\cos(2x) + B\sin(2x),\ \ A,\ \ B \in \mathbb{R}.$$

And since $3+2i$ is not a root of the characteristic equation, we seek a particular solution in the form:

$$y_p = e^{3x}(a\cos(2x) + b\sin(2x))$$

where a and b are constants to be determined. Substituting the expression of y_p into equation (6) and identifying terms, we obtain:

$$\begin{cases} 9a + 12b = 1 \\ 9b - 12a = 0 \end{cases} \Longleftrightarrow \begin{cases} a = \dfrac{1}{25} \\ b = \dfrac{4}{75} \end{cases}$$

Thus, the particular solution y_p can be written in the form:

$$y_p = e^{3x}\left(\frac{1}{25}\cos(2x) + \frac{4}{75}\sin(2x)\right)$$

The general solution of equation (6) is:

$$\boxed{y = A\cos(2x) + B\sin(2x) + e^{3x}\left(\frac{1}{25}\cos(2x) + \frac{4}{75}\sin(2x)\right),\ \ A,\ \ B \in \mathbb{R}}$$

Exercise 4:

We have

$$y'' + 2y' + 4y = xe^x \cdots\cdots (E)$$

1. Solvig the homogenuous equation $y''+2y'+4y = 0$. The characteristic equation is: $r^2 + 2r + 4 = 0$. It has two complex conjugate roots:

$$r_1 = -1 - i\sqrt{3}, \text{ and } r_2 = -1 + i\sqrt{3}$$

Thus, the solutions of the homogeneous equation are:

$$y_h = e^{-x}(C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x)),\ C_1,\ C_2 \in \mathbb{R}.$$

2. The particular solution y_p. Since $\alpha = 1$ is not a root of the characteristic equation, we seek a particular solution in the form:

$$y_p = (ax+b)e^x$$

so

$$\begin{aligned} y_p = (ax+b)e^x \quad &\Longleftrightarrow \quad y_p' = ae^x + (ax+b)e^x \\ &\Longleftrightarrow \quad y_p'' = 2ae^x + (ax+b)e^x \end{aligned}$$

Substituting the expression of y_p into equation (E) and identifying terms, we obtain:

$$(7ax + 4a + 7b)e^x = xe^x$$

We find, after identifying the coefficients:

$$a = \frac{1}{7},\ \text{and } b = -\frac{4}{49}$$

Thus, the particular solution y_p can be written in the form:

$$y_p = \left(\frac{1}{7}x - \frac{4}{49}\right)e^x$$

Then the generale solution is given by

$$y = y_p + y_h = e^{-x}\left(C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x)\right) + \left(\frac{1}{7}x - \frac{4}{49}\right)e^x,\ C_1,\ C_2 \in \mathbb{R}.$$

3. Let h be a solution of (E), satisfying $h(0) = 1$ and $h(1) = 0$. Let us define:

$$h(x) = e^{-x}\left(C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x)\right) + \left(\frac{1}{7}x - \frac{4}{49}\right)e^x,\ C_1,\ C_2 \in \mathbb{R}.$$

Then, we determine the values of C_1, and C_2

$$h(0) = 1,\ \text{and } h(1) = 0 \Longrightarrow C_1 = \frac{53}{49},\ \text{and } C_2 = -\frac{53\cos(\sqrt{3}) + 3e^2}{49\sin(\sqrt{3})}.$$