Solution of series $N°2$

Exercise 1:

Reminder 1:

If the function f is Riemann integrable on $[a,b]$ then the number $\int_a^b f(x)\,dx$ is the common limit of the two sequences:

$$u_n = \frac{b-a}{n}\sum_{k=0}^{n-1} f\left(a+\frac{b-a}{n}k\right),\ v_n = \frac{b-a}{n}\sum_{k=1}^{n} f\left(a+\frac{b-a}{n}k\right)$$

1. $\int_0^1 x^2\,dx$. We have

$$\begin{aligned}\int_0^1 x^2\,dx &= \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2}\\ &= \lim_{n\to+\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2\\ &= \lim_{n\to+\infty}\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{1}{3}.\end{aligned}$$

2. $\int_0^1 e^x\,dx$. So

$$\begin{aligned}\int_0^1 e^x\,dx &= \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}(e^{\frac{1}{n}})^k \quad \text{(sum of the geometric sequence)}\\ &= e-1.\end{aligned}$$

Exercise 2:

1. $u_n = n\sum_{k=0}^{n-1}\frac{1}{k^2+n^2}$. We have

$$\lim_{n\to+\infty} n\sum_{k=0}^{n-1}\frac{1}{k^2+n^2} = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\frac{k^2}{n^2}+1}$$

Suppose taht $f(x)=\frac{1}{x^2+1}$, $a=0$, $b=1$, so

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\frac{k^2}{n^2}+1} = \int_0^1\frac{1}{x^2+1}\,dx = \arctan\ x\,]_0^1 = \frac{\pi}{4}$$

2. $u_n = \displaystyle\sum_{k=0}^{n} \frac{n}{(n+k)^2}$. We have

$$\lim_{n\to+\infty} \sum_{k=0}^{n} \frac{n}{(n+k)^2} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n} \frac{1}{(1+\frac{k}{n})^2}$$

Suppose taht $f(x) = \dfrac{1}{x^2}$, $a = 1$, $b = 2$, then

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n} \frac{1}{(1+\frac{k}{n})^2} = \int_1^2 \frac{1}{x^2}\, dx = -\frac{1}{x}\,]_1^2 = \frac{1}{2}.$$

3. $u_n = \displaystyle\prod_{k=1}^{n} \left(1 + \frac{k^2}{n^2}\right)^{\frac{1}{n}}$. We have

$$\prod_{k=1}^{n} \left(1 + \frac{k^2}{n^2}\right)^{\frac{1}{n}} = exp\left[\frac{1}{n}\sum_{k=1}^{n} \ln\left(1 + \frac{k^2}{n^2}\right)\right]$$

so

$$\lim_{n\to\infty} u_n = exp\left[\lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=1}^{n} \ln\left(1 + \frac{k^2}{n^2}\right)\right)\right]$$

hence, $\displaystyle\lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=1}^{n} \ln\left(1 + \frac{k^2}{n^2}\right)\right) = \int_0^1 \ln(1+x^2)\, dx$ $(f(x) = \ln(1+x^2),\ a = 0,\ b = 1)$.

Using the integration by part, we get

$$\begin{cases} u' = 1 \\ v = \ln(1+x^2) \end{cases} \Longrightarrow \begin{cases} u = x \\ v' = \dfrac{2x}{1+x^2} \end{cases}$$

therefore, $\displaystyle\int_0^1 \ln(1+x^2)dx = \ln 2 - 2 + \frac{\pi}{2}$. Then

$$\lim_{n\to\infty} u_n = exp\left(\ln 2 - 2 + \frac{\pi}{2}\right) = 2exp\left(\frac{\pi}{2} - 2\right).$$

4. $u_n = \displaystyle\frac{1}{n}\sum_{k=0}^{n-1} \cos\left(\frac{k\pi}{2n}\right)$. We have

$$\begin{aligned} \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \cos\left(\frac{k\pi}{2n}\right) &= \frac{2}{\pi}\lim_{n\to+\infty} \frac{\pi}{2n}\sum_{k=0}^{n-1} \cos\left(0 + \frac{k}{n}\frac{\pi}{2}\right) \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x\, dx \end{aligned}$$

and $\int_0^{\frac{\pi}{2}} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}} = 1$. So

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \cos\left(\frac{k\pi}{2n}\right) = \frac{2}{\pi}.$$

Exercise 3:

1. It is clear that

$$u_n = \sum_{k=1}^{n} \frac{n+k}{n^2+k^2} = \sum_{k=1}^{n} \frac{1}{n}\frac{1+k/n}{1+(k/n)^2}$$

is a Riemann sum of the function $f(x) = \dfrac{1+x}{1+x^2}$, on the interval $[0,1]$. In consequence,

$$\lim_{n\to+\infty} u_n = \int_0^1 \frac{1+x}{1+x^2}\, dx = \left[\arctan(x) + \frac{\ln(1+x^2)}{2}\right]_0^1 = \frac{\pi + \ln(4)}{4}.$$

2. It is clear that

$$s_n = \sum_{k=n+1}^{2n} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k+n} = \sum_{k=1}^{n} \frac{1}{n}\frac{1}{1+k/n}$$

is a Riemann sum of the function $f(x) = \dfrac{1}{x}$, on the interval $[1,2]$. In consequence,

$$\lim_{n\to+\infty} s_n = \int_1^2 \frac{1}{x}\, dx = [\ln(x)]_1^2 = \ln(2).$$

3. It is clear that

$$v_n = \frac{1}{n^2}\sum_{k=1}^{n} k^\alpha \sin(k/n) = n^{\alpha-1}\sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right)^\alpha \sin(k/n)$$

is the product of $n^{\alpha-1}$ by a Riemann sum of the function $f(x) = x^\alpha \sin(x)$ on the interval $[0,1]$. In consequence,

$$\lim_{n\to+\infty} v_n.n^{\alpha-1} = \int_0^1 x^\alpha \sin(x)\, dx.$$

- If $\alpha > 1$, we thus conclude that v_n goes to $+\infty$.

- If $\alpha = 1$, then

$$\lim_{n\to+\infty} v_n = \int_0^1 x\sin(x)\ dx = [\sin(x) - x\cos(x)]_0^1 = \sin(1) - \cos(1).$$

- If $\alpha < 1$, then the integral $\int_0^1 x^\alpha \sin(x)\ dx$ is finite and we thus conclude that v_n goes to 0 as n goes to $+\infty$.

Exercise 4:

Reminder 2:

1.**Integration by parts:**

Suppose that $u,\ v : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable in $]a,b[$, and $u',\ v'$ are integrable on $[a,b]$. Then

$$\int_a^b uv'\ dx = [uv]_a^b - \int_a^b u'v\ dx.$$

2. **Integration by parts in indefinite integrals:**

Let I be a interval for $\mathbb{R}$ and $u,\ v$ are functions of class C^1 on the interval I then

$$\int\ uv'\ dx = [uv] - \int\ u'v\ dx + c.$$

1. $\int x^2\ln(\frac{x-1}{x})\ dx$. Suppose that:

$$\begin{cases} v' = x^2 \\ u = \ln\left(\dfrac{x-1}{x}\right) \end{cases} \Longrightarrow \begin{cases} v = \dfrac{1}{3}x^3 \\ u' = \dfrac{1}{x(x-1)} \end{cases}$$

So

$$\begin{aligned} I &= \frac{x^3}{3}\ \ln\left(\frac{x-1}{x}\right) - \int \frac{x^3}{3x(x-1)}\ dx \\ &= \frac{x^3}{3}\ \ln\left(\frac{x-1}{x}\right) - \frac{1}{3}\int \frac{x^2}{x-1}\ dx \end{aligned}$$

By Euclidean division we get:

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

So

$$\begin{aligned} I &= \frac{x^3}{3} \ln\left(\frac{x-1}{x}\right) - \frac{1}{3}\int \left(x + 1 + \frac{1}{x-1}\right) dx \\ &= \frac{x^3}{3} \ln\left(\frac{x-1}{x}\right) - \frac{x^2}{6} - \frac{x}{3} - \frac{1}{3}\ln|x-1| + C. \end{aligned}$$

2. $\int_0^1 x \arctan x \; dx$. Assume that:

$$\begin{cases} v' = x \\ u = \arctan x \end{cases} \Longrightarrow \begin{cases} v = \dfrac{x^2}{2} \\ u' = \dfrac{1}{1+x^2} \end{cases}$$

So

$$\begin{aligned} I &= \left[\frac{x^2}{2} \arctan x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2} \; dx \\ &= \frac{\pi}{8} - \frac{1}{2}\int_0^1 1 - \frac{1}{x^2+1} \; dx = \frac{\pi}{8} - \frac{1}{2}\left[x - \arctan x\right]_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2}. \end{aligned}$$

3. $\int \cos x e^x \; dx$. Using the integration by parts twice, we get

$$I = \frac{1}{2}e^x(\sin x + \cos x) + C.$$

Exercise 5:

Reminder 3:

1.**Change of variable:**

Let $\phi : [a,b] \to \mathbb{R}$ be a continuously differentiable function, let f be continuous over $\phi([a,b])$, then

$$\int_a^b f(x)\ dx = \int_\alpha^\beta f(\phi(t))\phi'(t)\ dt,$$

where $b = \phi(\beta),\ a = \phi(\alpha)$, and $x = \phi(t)$, $dx = \phi'(t)\ dt$.

2. **Change the variable in indefinite integrals:**

Let $h : I \to J$ of class C^1. We put $x = h(t)$ and $dx = h'(t)\ dt$ then

$$\int f(x)\ dx = (h(t))h'(t)\ dt,$$

where $t = h^{-1}(x)$.

1. $\displaystyle\int \frac{1}{3\sqrt[3]{x+1} - x + 1}\ dx$. We put $t = (x+1)^{\frac{1}{3}}$, then $x+1 = t^3 \Longrightarrow dx = 3t^2\ dt$, so

$$I = \int \frac{3t^2}{-t^3 + 3t + 2}\ dt = \int \frac{1}{3t - t^3 + 2} .3t^2\ dt.$$

We have $-t^3 + 3t + 2 = (2-t)(t+1)^2$. We put

$$\begin{aligned}\frac{3t^2}{3t - t^3 + 2} = \frac{3t^2}{(2-t)(t+1)^2} &= \frac{a}{2-t} + \frac{b}{t+1} + \frac{c}{(t+1)^2} \\ &= \frac{a(t+1)^2 + b(2-t)(t+1) + c(2-1)}{(2-t)(t+1)^2}\end{aligned}$$

and we get $a = \dfrac{4}{3},\ b = \dfrac{-5}{4},\ c = 1.$ so

$$\begin{aligned} I &= \frac{4}{3}\int \frac{1}{t-2}\ dt - \frac{5}{3}\int \frac{1}{t+1}\ dt + \int \frac{1}{(t+1)^2}\ dt \\ &= \frac{4}{3}\ln|t-2| - \frac{5}{3}\ln|t+1| - \frac{1}{t+1} + C \end{aligned}$$

Substituting $t = (x+1)^{\frac{1}{3}}$ we get:

$$\frac{4}{3}\ln\left|\sqrt[3]{x+1}-2\right| - \frac{5}{3}\ln\left|\sqrt[3]{x+1}+1\right| - \frac{1}{\sqrt[3]{x+1}+1} + C.$$

2. $\int_{-1}^{\frac{1}{2}} \sqrt{x^2+2x+5}\, dx$. We have $x^2+2x+5 = (x+1)^2+2^2$, we put $x+1 = 2sh(t)$, so $dx = 2ch(t)\, dt$. And

$$\begin{aligned} x &= \frac{1}{2} \Longrightarrow (-1+2sh(t) = -1) \Longleftrightarrow t = sh^{-1}\left(\frac{3}{4}\right) = \ln(2) \\ x &= -1 \Longrightarrow (-1+2sh(t) = -1) \Longleftrightarrow t = 0 \end{aligned}$$

Remark:

$$\begin{aligned} sh^{-1}(x) &= \ln\left(x+\sqrt{x^2+1}\right) \\ sh^{-1}\left(\frac{3}{4}\right) &= \ln\left(\frac{3}{4}+\sqrt{\left(\frac{3}{4}\right)^2+1}\right) \\ &= \ln\left(\frac{3}{4}+\frac{5}{4}\right) = \ln\left(\frac{8}{4}\right) = \ln(2). \end{aligned}$$

So

$$I = \int_{-1}^{\frac{1}{2}} \sqrt{x^2+2x+5}\ dx = \int_0^{\ln(2)} \sqrt{4sh^2(t)+4}.(2ch(t))\ dt = 4\int_0^{\ln(2)} \sqrt{ch^2(t)}.ch(t)\ dt$$

Since $t \in [0, \ln(2)]$, $ch(t) > 0$, then:

$$I = 4\int_0^{\ln(2)} ch^2(t)\ dt = 4\int_0^{\ln(2)} \frac{ch(2t)+1}{2}\ dt = [sh(2t)+2t]_0^{\ln(2)} = sh(2\ln(2)) + 2\ln(2)$$

Hence

$$I = 2\ln(2) + \frac{15}{8}.$$

3. $\int \frac{\sin x}{1+\sin x}\ dx$. We put $t = \tan\left(\frac{x}{2}\right)$, where $\cos(x) = \frac{1-t^2}{1+t^2}$, $\sin(x) = \frac{2t}{1+t^2}$, $dx = \frac{1}{1+t^2}\ dt$. Then

$$I=\int \frac{\sin x}{1+\sin x}\,dx=\int \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}}.\frac{2}{1+t^2}\,dt=\int \frac{4t}{(t^2+1)(1+t)^2}\,dt$$

and we put

$$\frac{4t}{(t^+1)(t+1)^2}=\frac{at+b}{t^2+1}+\frac{c}{t+1}+\frac{d}{(t+1)^2}$$

we get $a=0,\ b=2,\ c=0,\ d=-2$. So

$$I=2\int \frac{1}{t^2+1}\,dt-2\int \frac{1}{(t+1)^2}\,dt=2\arctan(t)+\frac{2}{t+1}+C$$

Substituting $t=\tan\left(\frac{x}{2}\right)$ we get:

$$I=x+\frac{2}{\tan\left(\frac{x}{2}\right)+1}+C.$$

4. $\int \frac{x}{\sqrt{x+1}}\,dx$. We put $t=x+1 \Longrightarrow dt=dx$, then

$$\begin{aligned}\int \frac{x}{\sqrt{x+1}}\,dx &= \int \frac{t-1}{\sqrt{t}}\,dt\\ &= \int \frac{t}{\sqrt{t}}-\frac{1}{\sqrt{t}}\,dt\\ &= \frac{2}{3}t^{\frac{3}{2}}-2\sqrt{t}+C\end{aligned}$$

Substituting $t=x+1$, we get

$$\frac{2}{3}(x+1)^{\frac{3}{2}}-2\sqrt{x+1}+C.$$

5. $\int \frac{1}{\sin x}\,dx$. We put $t=\tan\left(\frac{x}{2}\right) \Longrightarrow \sin x=\frac{2t}{t^2+1}$, and $\cos x=\frac{1-t^2}{t^2+1}$, $dx=\frac{2dt}{t^2+1}$, so

$$I=\int \frac{1+t^2}{2t}.\frac{2}{1+t^2}\,dt=\int \frac{1}{t}\,dt=\ln|t|+C$$

Substituting $t=\tan\left(\frac{x}{2}\right)$, we get

$$I = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C.$$

Exercise 6:

1. Observe that

$$\begin{aligned} I - J &= \int_0^{\frac{\pi}{2}} \frac{\cos^2(x) - \sin^2(x)}{\cos(x) + \sin(x)}\ dx = \int_0^{\frac{\pi}{2}} (\cos(x) - \sin(x))\ dx \\ &= [\sin(x) + \cos(x)]_0^{\frac{\pi}{2}} = \left(\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right) - (\sin(0) + \cos(0)) \\ &= 0. \end{aligned}$$

   So $I = J$.

2. We have

$$\begin{aligned} \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) &= \sqrt{2}\left(\cos(x)\cos\left(\frac{\pi}{4}\right) + \sin(x)\sin\left(\frac{\pi}{4}\right)\right) \\ &= \sqrt{2}\left(\frac{\sqrt{2}}{2}\cos(x) + \frac{\sqrt{2}}{2}\sin(x)\right) \\ &= \cos(x) + \sin(x). \end{aligned}$$

3. We note that

$$\begin{aligned} I + J &= \int_0^{\frac{\pi}{2}} \frac{\cos^2(x) + \sin^2(x)}{\cos(x) + \sin(x)}\ dx \\ &= \int_0^{\frac{\pi}{2}} \frac{dx}{\cos(x) + \sin(x)} \\ &= \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)} \end{aligned}$$

   We put $u = x - \frac{\pi}{4} \Longrightarrow dx = du$. Also

$$\begin{aligned} x = 0 &\Longrightarrow u = -\frac{\pi}{4} \\ x = \frac{\pi}{2} &\Longrightarrow u = \frac{\pi}{4} \end{aligned}$$

   So

$$I + J = \frac{1}{\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{du}{\cos(u)} = \frac{\sqrt{2}}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{du}{\cos(u)}.$$

4. For calculate $I+J$, we put $u=\tan\left(\frac{x}{2}\right)$, then we have

$$dx=\frac{2du}{1+u^2},\ \cos(x)=\frac{1-u^2}{1+u^2}$$

now, if $x=-\frac{\pi}{4}\Longrightarrow u=\tan\left(-\frac{\pi}{8}\right)$, and if $x=\frac{\pi}{4}\Longrightarrow u=\tan\left(\frac{\pi}{8}\right)$. Since

$$\cos\left(\frac{\theta}{2}\right)=\sqrt{\frac{1+\cos(\theta)}{2}},\ \sin\left(\frac{\theta}{2}\right)=\sqrt{\frac{1-\cos(\theta)}{2}}$$

then

$$\sin\left(\frac{\pi}{8}\right)=\sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)},\text{ and }\cos\left(\frac{\pi}{8}\right)=\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)}$$

so

$$\tan\left(\frac{\pi}{8}\right)=\frac{\sin(\pi/8)}{\cos(\pi/8)}=\sqrt{2}-1,\text{ and }\tan\left(-\frac{\pi}{8}\right)=-\sqrt{2}+1$$

Hence,

$$I+J=\frac{\sqrt{2}}{2}\int_{-\sqrt{2}+1}^{\sqrt{2}-1}\frac{\frac{2du}{1+u^2}}{\frac{1-u^2}{1+u^2}}=\sqrt{2}\int_{-\sqrt{2}+1}^{\sqrt{2}-1}\frac{du}{1-u^2}.$$

We have

$$\frac{1}{1-u^2}=\frac{a}{1-u}+\frac{b}{1+u}$$

we get $a=1/2$, and $b=1/2$. then

$$\begin{aligned}\sqrt{2}\int_{-\sqrt{2}+1}^{\sqrt{2}-1}\frac{du}{1-u^2} &= \frac{\sqrt{2}}{2}\left[\int_{-\sqrt{2}+1}^{\sqrt{2}-1}\frac{du}{1-u}+\int_{-\sqrt{2}+1}^{\sqrt{2}-1}\frac{du}{1+u}\right]\\ &= \frac{\sqrt{2}}{2}\left[-\ln(1-u)+\ln(1+u)\right]_{-\sqrt{2}+1}^{\sqrt{2}-1}\\ &= \frac{\sqrt{2}}{2}\left[2\ln(\sqrt{2})-2\ln(2-\sqrt{2})\right]\\ &= \sqrt{2}\ln\left(\frac{\sqrt{2}}{2-\sqrt{2}}\right)=\sqrt{2}\ln\left(\sqrt{2}+1\right).\end{aligned}$$

5. We have

$$\begin{cases} I - J &= 0 \\ I + J &= \sqrt{2}\ln\left(\sqrt{2}+1\right) \end{cases}$$

so $I = J = \dfrac{\sqrt{2}}{2}\ln\left(\sqrt{2}+1\right)$.

Exercise 7:

Reminder 3:

1.**Integration of the type** $J = \displaystyle\int \frac{1}{(1+t^2)^k\ dt}$**:**

$$\forall\ k \geq 1:\ 2kJ_{k+1} = (2k-1)J_k + \frac{t}{(1+t^2)^k};\ J_1 = \arctan(x) + C.$$

2. **Integration of the type** $\displaystyle\int R(\sin(x),\ \cos(x))\ dx$**:**

using the change in the variable $t = \tan\left(\dfrac{x}{2}\right)$, where:

$$\cos(x) = \frac{1-t^2}{1+t^2};\ \sin(x) = \frac{2t}{1+t^2};\ dx = \frac{2}{1+t^2}\ dt.$$

3. **Integration of the type** $\displaystyle\int R\left(x;\ \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}};\ \left(\frac{ax+b}{cx+d}\right)^{\frac{p}{q}};\cdots;\left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right)\ dx$**:**

use a change in the variable $t = \left(\dfrac{ax+b}{cx+d}\right)^{\frac{1}{k}}$, where k is the Least Common Multiple (LCM) of the numbers $n,\ q,\cdots,\ s$.

1. $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\ dx$. We use the formula

$$J_{k+1} = \frac{2k-1}{2k}J_k + \frac{x}{2k(1+x^2)^k}$$

where: $J_k = \displaystyle\int \frac{1}{(1+x^2)^k}\ dx$, and $J_1 = \displaystyle\int \frac{1}{1+x^2}\ dx = \arctan(x)$. So for $k = 1$ we obtain:

$$\begin{aligned} J_2 &= \int_0^1 \frac{1}{(1+x^2)^2}\ dx = \left.\frac{x}{2(1+x^2)}\right]_0^1 + \frac{1}{2}J_1]_0^1 \\ &= \frac{1}{4} + \frac{1}{2}\arctan(x)]_0^1 \\ &= \frac{1}{4} + \frac{1}{2}\frac{\pi}{4} = \frac{1}{4} + \frac{\pi}{8}. \end{aligned}$$

2. $\displaystyle\int_{-1}^{1} \frac{1}{x^2+4x+7}\,dx$. We have

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2+4x+7}\,dx &= \int_{-1}^{1} \frac{1}{(x+2)^2+3}\,dx \\ &= \frac{1}{3}\int_{-1}^{1} \frac{1}{\left(\frac{x+2}{\sqrt{3}}\right)^2+1}\,dx\end{aligned}$$

We use the change of variable $t = \dfrac{x+2}{\sqrt{3}}$, so

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2+4x+7}\,dx &= \int_{-1}^{1} \frac{1}{(x+2)^2+3}\,dx \\ &= \frac{\sqrt{3}}{3}\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{dt}{t^2+1} = \frac{\sqrt{3}}{3}\arctan(t)\Big]_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \\ &= \frac{\sqrt{3}}{3}\left[\arctan\left(\sqrt{3}\right) - \arctan\left(\frac{\sqrt{3}}{3}\right)\right].\end{aligned}$$

3. $\displaystyle\int_{0}^{1} \frac{3x+1}{(1+x)^2}\,dx$. We use the change of variable $t = x+1 \Longrightarrow x = t-1$, so

$$\begin{aligned}I &= \int_{1}^{2} \frac{3(t-1)+1}{t^2}\,dt \\ &= \int_{1}^{2} \frac{3t}{t^2}\,dt - \int_{1}^{2} \frac{2}{t^2}\,dt \\ &= 3\ln t]_1^2 + \frac{2}{t}\Big]_1^2 \\ &= 3\ln 2 + 2\left(\frac{1}{2}-1\right) = 3\ln 2 - 1.\end{aligned}$$

4. $\displaystyle\int \frac{dx}{\sin^3 x}$. We pose $t = \tan\left(\frac{x}{2}\right) \Longrightarrow dx = \dfrac{2dt}{1+t^2}$, and $\sin x = \dfrac{2t}{1+t^2}$. So

$$\begin{aligned}
\int \frac{dx}{\sin^3 x} &= \frac{1}{4}\int \frac{(1+t^2)^2}{t^3}\,dt \\
&= \frac{1}{4}\int \left(\frac{1}{t^3}+\frac{2}{t}+t\right)\,dt \\
&= \frac{1}{4}\left(-\frac{1}{2t^2}+2\ln t+\frac{t^2}{2}\right) \\
&= \frac{1}{4}\left(-\frac{1}{2\tan^2(\frac{x}{2})}+2\ln\left(\tan(\tfrac{x}{2})\right)+\frac{\tan^2(\frac{x}{2})}{2}\right).
\end{aligned}$$

5. $\displaystyle\int \frac{\sin 2x}{\sin^2 x-5\sin x+6}\,dx$. We pose $y=\sin x \Longrightarrow dy=\cos x\,dx$, so

$$\begin{aligned}
\int \frac{\sin 2x}{\sin^2 x-5\sin x+6}\,dx &= \int \frac{2\sin x\cos x}{\sin^2 x-5\sin x+6}\,dx \\
&= \int \frac{2y}{y^2-5y+6}\,dy
\end{aligned}$$

where

$$\frac{2y}{y^2-5y+6}=\frac{2y}{(y-3)(y-2)}=\frac{a}{y-3}+\frac{b}{y-2}$$

After a simple calculation, we find $a=6$ and $b=4$. As a result

$$\begin{aligned}
\int \frac{2y}{y^2-5y+6}\,dy &= \int\left(\frac{6}{y-3}-\frac{4}{y-2}\right)\,dy \\
&= 6\ln|y-3|-4\ln|y-2|+C \\
&= 6\ln|\sin x-3|-4\ln|\sin x-2|+C
\end{aligned}$$