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Chapter 1

Differential Equations

Calculus is the mathematics of change, and rates of change are expressed through derivatives. Thus, one of the most common ways to use calculus is to set up an equation containing an unknown function $y = f(x)$ and its derivative, known as a differential equation. Solving such equations often provides information about how quantities change and frequently provides insight into how and why the changes occur.

1.1 Differential equations of order n

Definition 1.1.1. *Let $n \in \mathbb{N},\ n \geq 1$, we call a differential equation of order n and of unknown function y any relation of the form*

$$y^{(n)} = f(x, y(x), y'(x), \cdots, y^{(n-1)}(x)) \tag{1.1}$$

with initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1} \tag{1.2}$$

where f is a function defined on a part of $\mathbb{R}^{n+1}$, $(x_0, y_0, \cdots, y_{n-1})$ is vector fixed in $\mathbb{R}^{n+1}$ and the unknown is a function y of class C^n defined on an open interval of $\mathbb{R}$ containing x_0.

- *We call solution of this equation any function y of class C^n defined on an open interval containing x_0 and verifying the equations (1.1), and (1.2).*

- *The largest order of derivation is called the order of equation (1.1).*

Example 1.1.2. *1. $y' = y + x$ is a first order differential equation.*

2. $y'' - 3y' + 2y = 0$ is a second order differential equation.

1.2 First Order Differential Equations

Let $f : D \subset \mathbb{R}^2 \Longrightarrow \mathbb{R}$ and $y : I \subset \mathbb{R} \Longrightarrow \mathbb{R}$ two functions such that y is differentiable on I.

Definition 1.2.1. *We call a first order differential equation any equation of type*

$$y'(x) = f(x, y) \tag{1.3}$$

We say that (1.3) admits a solution $y_0(x)$ if $y_0' = f(x, y_0)$.

1.2.1 Equations with separable variables

A first order differential equation is said to have separate variables if it is of the form

$$f(y)dy = g(x)dx \tag{1.4}$$

where $f,\ g$ are two real (continuous) functions of the real variable. To solve (1.4) we use the definition

$$y' = \frac{dy}{dx}$$

hence

$$\begin{aligned}
f(y)y' &= g(x) \Longrightarrow f(y)\frac{dy}{dx} = g(x)\\
&\Longrightarrow f(y)dy = g(x)dx\\
&\Longrightarrow \int f(y)dy = \int g(x)dx\\
&\Longrightarrow F(x) = G(x) + c
\end{aligned}$$

where F is the antiderivative of f and G is the antiderivative of g. Moreover, if F admits an inverse function F^{-1} then

$$y(x) = F^{-1}(G(x) + c).$$

Example 1.2.2. *Let's solve the equation* $\frac{1}{y}y' = (x^2 + 1)$.

$$\begin{aligned}
\frac{1}{y}y' &= (x^2+1) \Longrightarrow \frac{1}{y}\frac{dy}{dx} = (x^2+1)\\
&\Longrightarrow \int \frac{1}{y}dy = \int (x^2+1)\,dx\\
&\Longrightarrow \ln|y| = \frac{x^3}{3} + x + c, \quad c \in \mathbb{R}\\
&\Longrightarrow |y| = e^{\frac{x^3}{3}+x+c}, \quad c \in \mathbb{R}\\
&\Longrightarrow |y| = ke^{\frac{x^3}{3}+x}, \quad k = e^c\\
&\Longrightarrow y = Ke^{\frac{x^3}{3}+x}, \quad K \in \mathbb{R}^*, \quad K = \pm k.
\end{aligned}$$

Example 1.2.3. $y'(x) = x^2y(x) + x^2$, *is with separable variables. Indeed, we can bring it back to the form*

$$\frac{y'(x)}{y(x)+1} = x^2$$

consequently, passing to the antiderivatives, we have

$$\ln|y+1| = \frac{x^3}{3} + c, \quad c \in \mathbb{R}$$

which implies to

$$y(x) = Ke^{\frac{x^3}{3}} - 1, \quad K = \pm e^c$$

1.2.2 Homogeneous Differential Equations

A first order differential equation is said to be homogeneous with respect to x and y if it is of the form

$$y' = f\left(\frac{y}{x}\right) \tag{1.5}$$

where f is a real (continuous) function of the real variable.

To solve (1.5) we put

$$t = \frac{y}{x}$$

so $y = xt$, and then $dy = x\ dt + t\ dx$. we replace in the equation (1.5), we obtain

$$\begin{aligned} \frac{dy}{dx} &= f\left(\frac{y}{x}\right) \Longrightarrow dy = f(t)\ dx \\ &\Longrightarrow x\ dt + t\ dx = f(t)\ dx \\ &\Longrightarrow x\ dt = (f(t) - t)\ dx \\ &\Longrightarrow \frac{dt}{f(t) - t} = \frac{dx}{x} \end{aligned}$$

this last equation is an equation with separate variables.

Example 1.2.4. *To solve the equation*

$$x^2y' = x^2 + y^2 - xy$$

by dividing both sides of the equality by x^2, this equation can be reduced to

$$y' = 1 + \left(\frac{y}{x}\right)^2 - \frac{y}{x}$$

we put $t = \dfrac{y}{x}$, then

$$\begin{aligned}
\frac{dy}{dx} &= 1+\left(\frac{y}{x}\right)^2-\frac{y}{x} = 1+t^2-t \\
&\Longrightarrow dy = (1+t^2-t)dx \\
&\Longrightarrow x\ dt + t\ dx = (1+t^2-t)\ dx \\
&\Longrightarrow x\ dt = (1+t^2-2t)\ dx \\
&\Longrightarrow \frac{dt}{1+t^2-2t} = \frac{dx}{x} \quad (t \neq 1) \\
&\Longrightarrow \int \frac{dt}{(1-t)^2} = \int \frac{dx}{x} \\
&\Longrightarrow \frac{1}{1-t} = \ln|x| + c
\end{aligned}$$

and so

$$t = 1 - \frac{1}{\ln|x|+c}$$

and we return to y

$$y = xt \Longrightarrow y = x - \frac{x}{\ln|x|+c}$$

and do not forget that if $t=1$ *then* $y=x$ *is also a solution of the given equation.*

Exercise 1.2.5. *Solve the differential equation:* $(2x+y)dx-(4x-y)dy=0$.

1.2.3 Linear Differential Equations

A first order differential equation is said to be linear if it has the form

$$a(x)y' + b(x)y = c(x) \tag{1.6}$$

where a, b and c are real (continuous) functions of the real variable, a being assumed not identically zero.

Equation (1.6) is called an equation with a second member, we associate it with the following equation called without a second member (or homogeneous equation)

$$a(x)y' + b(x)y = 0 \qquad\qquad (H.E)$$

1. **Solving the homogeneous equation associated:**

   to solve this one just need to separate the variables

$$\begin{aligned}\frac{y'}{y} &= -\frac{b(x)}{a(x)} \Longrightarrow \frac{dy}{y} = -\frac{b(x)}{a(x)}\,dx, \quad if \quad y \neq 0\\ &\Longrightarrow \int \frac{dy}{y} = \int -\frac{b(x)}{a(x)}\,dx\\ &\Longrightarrow \ln|y| = -A(x) + c, \quad c \in \mathbb{R}\end{aligned}$$

   where $\int -\frac{b(x)}{a(x)}\,dx = -A(x) + c$. So

$$y_h(x) = Ke^{-A(x)}, \quad K = \pm e^c$$

2. **Particular solution by variation of the constant:**

   To find y_p the particular solution of (1.6) we use the method of variation of the constant.

   We pose $y_p = K(x)e^{-A(x)}$, with K a function to be determined.

   and consequently

$$y'_p = K'(x)e^{-A(x)} - A'(x)K(x)e^{-A(x)} = K'(x)e^{-A(x)} - K(x)\frac{b(x)}{a(x)}e^{-A(x)}$$

   so

$$a(x)y'_p + b(x)y(x) = c(x)$$

   implies that

$$a(x)\left(K'(x)e^{-A(x)} - K(x)\frac{b(x)}{a(x)}e^{-A(x)}\right) + b(x)K(x)e^{-A(x)} = c(x)$$

   which give

$$a(x)K'(x)e^{-A(x)} = c(x)$$

   and so $K'(x) = \frac{c(x)}{a(x)e^{-A(x)}}$, hence $K(x) = \int \frac{c(x)}{a(x)e^{-A(x)}}\,dx$. So the particular solution is:

$$y_p = e^{-A(x)} \int \frac{c(x)}{a(x)e^{-A(x)}}\ dx$$

and therefore the general solution of (1.6) is given by

$$y = y_h + y_p = Ke^{-A(x)} + K(x)e^{-A(x)}, \quad k \in \mathbb{R}.$$

Example 1.2.6. *Solve the differential equation*

$$x^2y' + 2xy = e^x$$

we solve the homogeneous equation

$$\begin{aligned} x^2y' + 2xy &= 0 \Longrightarrow \frac{y'}{y} = -\frac{2x}{x^2} \\ &\Longrightarrow \frac{dy}{y} = -\frac{2x}{x^2}\ dx \\ &\Longrightarrow \ln|y| = -\ln x^2 + c, \quad c \in \mathbb{R} \\ &\Longrightarrow y = \frac{k}{x^2}, \quad k = \pm e^c \end{aligned}$$

and so the homogemeous equation is given by

$$y_h = \frac{k}{x^2}, \quad k \in \mathbb{R}.$$

Let's look for the particular solution:

$$y_p = \frac{k(x)}{x^2}$$

therefore

$$y_p' = \frac{k'(x)x^2 - 2xk(x)}{x^4}$$

we replace in the equation $x^2y' + 2xy = e^x$ *we obtain*

$$x^2\frac{k'(x)x^2 - 2xk(x)}{x^4} + 2x\frac{k(x)}{x^2} = e^x$$

which give $k'(x) = e^x$, and so $k(x) = e^x$. hence

$$y_p = \frac{e^x}{x^2}$$

then the general solution is given by

$$y = y_h + y_p = \frac{k + e^x}{x^2}, \quad k \in \mathbb{R}.$$

Exercise 1.2.7. *Solve the differential equation on $I = \left]0, \frac{\pi}{2}\right[$*

$$y' \sin(x) - y\cos(x) = x.$$

1.2.4 Bernoulli Equations

A first order differential equation is said to be Bernoulli if it has the form

$$a(x)y' + b(x)y + c(x)y^\alpha = 0 \tag{1.7}$$

where $a,\ b$ and c are real (continuous) functions of the real variable, α is a non-zero real and $\alpha \neq 1$, (if $\alpha = 1$, or $\alpha = 0$ the equation is linear).

To solve the equation (1.7) we will follow the following process

$$a(x)y' + b(x)y + c(x)y^\alpha = 0 \implies a(x)\frac{y'}{y^\alpha} + b(x)\frac{y}{y^\alpha} + c(x) = 0$$

$$\implies a(x)y'y^\alpha + b(x)y^{1-\alpha} + c(x) = 0$$

we pose $t = y^{1-\alpha}$, and then $t' = (1-\alpha)y'y^{-\alpha}$, we substitute in our equation to obtain

$$\frac{a(x)}{(1-\alpha)}t' + b(x)t + c(x) = 0$$

and this last equation is a linear equation that we know how to solve.

Example 1.2.8. *Solving the Bernoulli equation*

$$xy' - y = 2xy^2, \quad \alpha = 2 \qquad (EB)$$

(EB)$\Longrightarrow xy'y^{-2} - y^{-1} = 2x$. *We pose* $t = y^{-1} = \dfrac{1}{y} \Longrightarrow t' = -\dfrac{y'}{y^2} = -y'y^{-2}$.

so

$$(EB) \Longrightarrow -xt' - t = 2x$$

the homogeneous equation associated with (EB) is

$$\begin{aligned} -xt' - t = 0 &\Longrightarrow -xt' = t \\ &\Longrightarrow \frac{t'}{t} = -\frac{1}{x} \\ &\Longrightarrow \int \frac{t'}{t}\, dx = -\int \frac{1}{x}\, dx \\ &\Longrightarrow \ln|z| = -\ln|x| + c, \quad c \in \mathbb{R} \\ &\Longrightarrow t = e^{\ln\frac{1}{|x|}+c} = k\frac{1}{|x|}, \quad k = \pm e^{c}. \end{aligned}$$

The particular solution of (EB):

Note that $t = -x$ *is a particular solution of (EB), so the general solution is*

$$\begin{aligned} t &= \frac{k}{|x|} - x \\ t = \frac{1}{y} &\Longrightarrow y = \frac{1}{t} \Longrightarrow y = \frac{1}{\dfrac{k}{|x|} - x} = \frac{|x|}{k - x\,|x|}. \end{aligned}$$

1.2.5 Riccati Equation

A first order differential equation is said to be Riccati if it is of the form

$$a(x)y' + b(x)y + c(x)y^2 = d(x) \tag{1.8}$$

where $a,\ b,\ c$ and d are real (continuous) functions of the real variable, α being assumed to be non-zero. To solve (1.8) you must first find a particular solution y_p.

$$a(x)y_p' + b(x)y_p + c(x)y_p^2 = d(x)$$

We pose $t = y - y_p \Longrightarrow y = t + y_p \Longrightarrow y' = t' + y_p'$, and $y^2 = t^2 + y_p^2 + 2ty_p$. To determine t let's replace y in the equation

$$\begin{aligned} a(x)y' + b(x)y + c(x)y^2 = d(x) &\Longrightarrow a(x)(y_p' + t') + b(x)(y_p + t) + c(x)(y_p + t)^2 = d(x) \\ &\Longrightarrow a(x)t' + [b(x) + 2c(x)y_p]\, t + c(x)t^2 = 0 \end{aligned}$$

and this last equation is a Bernoulli equation in t with $\alpha = 2$, an equation that we know how to solve.

Example 1.2.9. *Solve the equation*

$$\begin{cases} e^{-x}y' - 2e^x y + y^2 = 1 - e^2 x \\ y_p = e^x \end{cases} \qquad \textit{(E.R)}$$

we pose $t = y - y_p \Longrightarrow y = t + y_p = t + e^x \Longrightarrow y' = t' + e^x$, *and* $y^2 = t^2 + e^{2x} + 2te^x$.

Replacing in (E.R) we obtain:

$$\begin{aligned} (E.R) &\Longleftrightarrow e^{-x}(t' + e^x) - 2e^x(t + e^x) + t^2 + 2te^x + e^{2x} = 1 - e^{2x} \\ &\Longleftrightarrow e^{-x}t' + t^2 = 0 \\ &\Longleftrightarrow e^{-x}t' = -t^2 \\ &\Longleftrightarrow \frac{t'}{t^2} = -e^x \\ &\Longleftrightarrow \int \frac{t'}{t^2}\, dx = \int -e^x\, dx \\ &\Longleftrightarrow \frac{-1}{t} = -e^x + c, \quad c \in \mathbb{R} \\ &\Longleftrightarrow t = \frac{1}{e^x - c} \end{aligned}$$

Then the general solution of (E.R) is

$$y = t + e^x \Longrightarrow y = \frac{1}{e^x - c} + e^x.$$

1.3 Second Order Differential Equations with Constant Coefficients

A second order linear differential equation with constant coefficients is a differential equation of the form:

$$ay'' + by' + cy = f(x) \qquad (E)$$

where $a,\ b$, and c are real constants $a \neq 0$, and $f \in C^0(I)$ a real function of the real variable.

1.3.1 The homogeneous equation associated with (E)(without second member)

$$ay'' + by' + cy = 0 \qquad (H.E)$$

Assume that $y = e^{rx}$ is solution of $(H.E)$ where r is constante

$$y' = re^{rx}, \quad and \quad y'' = r^2e^{rx}$$

. To solve (H.E) we substitute $y,\ y'$ and y'' into $(H.E)$, and we associate the algebraic equation called characteristic equation

$$ar^2 + br + c = 0$$

- 1^{st} **case:** $\Delta = b^2 - 4ac > 0$

  In this case the characteristic equation has two real roots

  $$r_1 = \frac{-b - \sqrt{\Delta}}{2a}, \qquad \text{and} \qquad r_1 = \frac{-b + \sqrt{\Delta}}{2a}$$

  then the general solution of (H.E) is given by the formula

  $$y_h = C_1e^{r_1x} + C_2e^{r_2x}, \quad C_1,\ C_2 \in \mathbb{R}.$$

- 2^{nd} **case:** $\Delta = b^2 - 4ac = 0$

  In this case the characteristic equation has a double real root

  $$r_0 = \frac{-b}{2a}$$

then the general solution of (H.E) is given by the formula

$$y_h = (C_1 + C_2 x)\, e^{r_0 x}, \quad C_1,\ C_2 \in \mathbb{R}.$$

- 3^{rd} **case:** $\Delta = b^2 - 4ac < 0$

  In this case the characteristic equation has two conjugated complex roots

$$r_1 = \alpha + i\beta, \qquad \text{and} \qquad r_2 = \alpha - i\beta$$

  then the general solution of (H.E) is given by the formula

$$y_0 = (C_1 \cos(\beta x) + C_2 \sin(\beta x))\, e^{\alpha x}, \quad C_1,\ C_2 \in \mathbb{R}.$$

Example 1.3.1. *To solve the equation*

$$y'' + y' + y = 0$$

we consider the characteristic equation

$$r^2 + r + 1 = 0 \Longrightarrow \Delta = -3 < 0$$

we have two complex roots conjugate:

$$r_1 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \qquad \textit{and} \qquad \frac{-1}{2} - i\frac{\sqrt{3}}{2}$$

and the general solution is given by

$$y_h = \left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x \right) e^{\frac{-1}{2}x}, \quad C_1,\ C_2 \in \mathbb{R}.$$

1.3.2 Solving the Linear Differential Equation of Order 2 with Second Member

A linear differential equation of order 2 is of the form $ay'' + by' + cy = f(x)$ this equation with second member solving in two steps:

1. We first solve the equation without a second member and obtain y_h.

2. We solve the equation with right hand side by looking for a particular solution y_p of (E) and in this case $y = y_h + y_p$.

Proposition 1.3.2. ***(Particular solution of the equation with second member)*** *Based on the expression of the second member, we will summarize the particular solutions possible in the case of the differential equation with second member:*

- *If $f(x) = P_n(x)$, we are looking for a particular solution of the form $x^k P_n(x) = Q_n(x)$*

  - $k = 0$ *if r is not solution of the characteristic equation.*
  - $k = 1$ *if r is a root of the characteristic equation.*
  - $k = 2$ *if r is a double root of the characteristic equation.*

  Only $y_p = Q_n(x)$, Q_n is a polynomial of degree n.

- *If $f(x) = e^{rx}(\lambda \cos\theta x + \mu \sin\theta x)$, $\quad \theta \in \mathbb{R}^*$*

  - *If $r + i\theta$, and $r - i\theta$ are not roots of (C.E), we are looking for a particular solution of the form*

  $$y_p = e^{rx}(\alpha \cos\theta x + \beta \sin\theta x).$$

  - *If $r + i\theta$, and $r - i\theta$ are roots of (C.E), then*

  $$y_p = xe^{rx}(A\cos\theta x + B\sin\theta x).$$

- *If $f(x) = e^{\alpha x} P_n(x)$, then $y_p = x^k e^{\alpha x} P_n(x)$*

  - $k = 0$ *if α is not solution of the characteristic equation.*
  - $k = 1$ *if α is a root of the characteristic equation.*
  - $k = 2$ *if r is a double root of the characteristic equation.*

Exercise 1.3.3. *Solve the following differential equations*

1. $y' \sin x = y \cos x$.
2. $y^2 + (x+1)y' = 0$.
3. $xy' - \alpha y = 0, \quad \alpha \in \mathbb{R}^*$.
4. $xy^2 y' = x^3 + y^3$.
5. $y'' + y = (x+1)$.
6. $y'' + 2y' + y = e^3 x$.
7. $y'' + 5y' + 6y = (x^2+1)$.