Abdelhafid Boussouf University Center, Mila
Institute of Mathematics and Computer Sciences
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Algebra II, Worksheet 1 answers

Exercise n°1 :

Solution : We know that $(E^X, +, \cdot)$ is a $K-$vector space if and only if it satisfies the following axioms :

(a_1) $(E^X, +)$ is a commutative group.
$(a_2) \forall \alpha, \beta \in K, \forall f \in E^X : (\alpha + \beta) \cdot f = \alpha \cdot f + \beta \cdot f$.
$(a_3) \forall \alpha \in K, \forall f, g \in E^X : \alpha \cdot (f + g) = \alpha \cdot f + \alpha \cdot g$.
$(a_4) \forall \alpha, \beta \in K, \forall f \in E^X : (\alpha \cdot \beta) \cdot f = \alpha \cdot (\beta \cdot f)$.
$(a_5) \forall f \in E^X : 1_K \cdot f = f$.

For (a_1) : According to the assumption, $(E^X, +)$ is a commutative group. Thus, (a_1) is verified.

For (a_2) : Since the mappings $(\alpha + \beta) \cdot f$ and $\alpha \cdot f + \beta \cdot f$ have the same domain X and the same codomain E, it suffices to show that

$$\forall x \in X : ((\alpha + \beta) \cdot f)(x) = (\alpha \cdot f + \beta \cdot f)(x).$$

Let $x \in X$, then

$$\begin{aligned} ((\alpha + \beta) \cdot f)(x) &= (\alpha + \beta) \cdot f(x) \text{ (by definition of scalar multiplication)} \\ &= \alpha \cdot f(x) + \beta \cdot f(x) \text{ (by distributivity in the field } K) \\ &= (\alpha \cdot f)(x) + (\beta \cdot f)(x) \text{ (by definition of scalar multiplication on mappings)} \\ &= (\alpha \cdot f + \beta \cdot f)(x) \text{ (by definition of addition of mappings)} \end{aligned}$$

Thus,

$$(\alpha + \beta) \cdot f = \alpha \cdot f + \beta \cdot f.$$

Hence, (a_2) is verified. The remaining axioms (a_3), (a_4), and (a_5) can be verified similarly by evaluating both sides of the equations at an arbitrary $x \in X$ and using the properties of the field K and the operations on mappings.

Exercise n°2 :

Solution : We can prove these rules using the axioms of a vector space. Let $x, y \in E, \alpha, \beta \in K$. Then

(1)

$$\alpha \cdot 0_E = \alpha \cdot (0_E + 0_E) = \alpha \cdot 0_E + \alpha \cdot 0_E$$

Subtracting $\alpha \cdot 0_E$ from both sides,

$$\alpha \cdot 0_E - \alpha \cdot 0_E = \alpha \cdot 0_E + \alpha \cdot 0_E - \alpha \cdot 0_E \implies 0_E = \alpha \cdot 0_E$$

and

$$0_K \cdot x = (0_K + 0_K) \cdot x = 0_K \cdot x + 0_K \cdot x$$

Subtracting $0_K \cdot x$ from both sides

$$0_K \cdot x - 0_K \cdot x = 0_K \cdot x + 0_K \cdot x - 0_K \cdot x \Longrightarrow 0_E = 0_K \cdot x$$

2. (j) $\Longleftarrow$: The converse implication follows directly from the first two properties.
* If $\alpha = 0_K$, then $\alpha \cdot x = 0_E$ by (1).
* If $x = 0_E$, then $\alpha \cdot x = 0_E$ by (1).

(j)$\Longrightarrow$: Let us prove the direct implication. Suppose that

$$\alpha \cdot x = 0_E.$$

• If $\alpha = 0_{\mathbb{K}}$, then there is nothing to prove (or if $\alpha = 0_{\mathbb{K}}$, we indeed have $\alpha = 0_{\mathbb{K}}$ or $x = 0_E$)
• If $\alpha \neq 0_{\mathbb{K}}$, then α is invertible in the field $\mathbb{K}$. Multiplying both sides of the equality by α^{-1}, we get

$$\alpha^{-1} \cdot (\alpha \cdot x) = \alpha^{-1} \cdot 0_E$$
$$\Longrightarrow \left(\alpha^{-1} \cdot \alpha\right) \cdot x = 0_E$$
$$\Longrightarrow 1_{\mathbb{K}} \cdot x = 0_E.$$
$$\Longrightarrow x = 0_E.$$

Now, suppose that $x \neq 0_E$. Then, we necessarily have $\alpha = 0_{\mathbb{K}}$, because otherwise (as above) we could multiply the equality $\alpha \cdot x = 0_E$ by α^{-1}, which would imply $x = 0_E$, contradicting our hypothesis.
(3). Using the distributive property of scalar multiplication over vector addition,

$$\alpha \cdot x = \alpha \cdot [(x - y) + y] = \alpha \cdot (x - y) + \alpha \cdot y$$

Subtracting $\alpha \cdot y$ from both sides

$$\alpha \cdot x - \alpha \cdot y = \alpha \cdot (x - y) + \alpha \cdot y - \alpha \cdot y = \alpha \cdot (x - y).$$

(4). We have

$$\alpha \cdot x = [(\alpha - \beta) + \beta] \cdot x = (\alpha - \beta) \cdot x + \beta \cdot x$$
$$\Longrightarrow \alpha \cdot x - \beta \cdot x = (\alpha - \beta) \cdot x + \beta \cdot x - \beta \cdot x = (\alpha - \beta) \cdot x.$$

Exercise n°3 :
Solution :
(a_1) $\mathbb{R}^2$, equipped with these two operations, is not an $\mathbb{R}-$vector space because the commutativity axiom of addition is not satisfied. Indeed,

$$(a, b) + (c, d) = (a + b, b + d) \text{ and } (c, d) + (a, b) = (c + d, d + b)$$

For example, choosing

$$(a, b) = (1, 2) \text{ and } (c, d) = (3, 1)$$

we obtain

$$1 + 2 \neq 3 + 1$$

(a_2) $\mathbb{R}^2$, equipped with these two operations, is not an $\mathbb{R}$-vector space because axiom (2) is not satisfied. Indeed, let $\alpha, \beta \in \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$, we have

$$(\alpha + \beta) \cdot (a, b) = ((\alpha + \beta)^2 a, (\alpha + \beta)^2 b)$$

On the other hand

$$\alpha \cdot (a, b) + \beta \cdot (a, b) = \left(\left(\alpha^2 + \beta^2\right) a, \left(\alpha^2 + \beta^2\right) b\right)$$

Since

$$(\alpha + \beta)^2 \neq \left(\alpha^2 + \beta^2\right), \text{(in general, except } \alpha = 0 \text{ or } \beta = 0)$$

This shows that the required axiom is not satisfied.
(a_3) $\mathbb{R}^2$ equipped with these two operations is not a $\mathbb{R}$-vector space because the commutativity axiom for the addition operation + is not satisfied. Indeed,

$$(a, b) + (c, d) = (c, d) \text{ and } (c, d) + (a, b) = (a, b)$$

For example, choosing

$$(a, b) = (1, 2) \text{ and } (c, d) = (3, 1)$$

we obtain

$$(3, 1) \neq (1, 2).$$

This shows that addition is not commutative under the given operation.

Exercise n°4 :

Solution : We know that

$$F \text{ is a subspace of } E \iff$$

$$\iff \begin{cases} \text{(a) } 0_E \in F \ (F \neq \phi) \ (F \text{ contains the zero vector of } E) \\ \text{(b) } F \text{ is closed under the internal law (addition) }, \forall x, y \in F : x + y \in F \\ \text{(c) } F \text{ is closed under the external law (scalar multiplication) } \forall \lambda \in \mathbb{K}, \forall x \in F : \lambda \cdot x \in F. \end{cases}$$

$$\iff \begin{cases} \text{(a) } 0_E \in F \\ \text{(b) } F \text{ is closed under linear combinations, } \forall \alpha, \beta \in \mathbb{K}, \forall x, y \in F, : \alpha \cdot x + \beta \cdot y \in F. \end{cases}$$

For F_1 **:** Let $a \in \mathbb{R}$, then we distinguish the following cases :

(a) * If $a \neq 0$, then $0_{\mathbb{R}^3} = (0, 0, 0) \notin F_1$, which implies that F_1 is not a subspace of $\mathbb{R}^3$.

* If $a = 0$, then F_1 becomes

$$F_1 = \{(x, y, z) \in E : x + y + z = 0\}.$$

It is clear that

$$0_{\mathbb{R}^3} = (0, 0, 0) \in F_1.$$

b) Let $X_1 = (x_1, y_1, z_1), X_2 \in (x_2, y_2, z_2) \in F_1$ and $\alpha, \beta \in \mathbb{R}$. Then, from the definition of F_1, we know

$$\begin{cases} x_1 + y_1 + z_1 = 0 \\ x_2 + y_2 + z_2 = 0 \end{cases} \quad ....(*)$$

On the other hand, we have

$$\alpha \cdot X_1 + \beta \cdot X_2 = (\alpha \cdot x_1 + \beta \cdot x_2, \alpha \cdot y_1 + \beta \cdot y_2, \alpha \cdot z_1 + \beta \cdot z_2)$$

We get

$$(\alpha \cdot x_1 + \beta \cdot x_2) + (\alpha \cdot y_1 + \beta \cdot y_2) + \alpha \cdot z_1 + \beta \cdot z_2) =$$

$$= \alpha\,(x_1 + y_1 + z_1) + \beta\,(x_2 + y_2 + z_2) \overset{(*)}{=} 0$$

Therefore, $\alpha \cdot X_1 + \beta \cdot X_2 \in F_1$, which shows that$F_1$ is indeed a subspace of $\mathbb{R}^3$.

For F_2 **: (the set of even mappings) :**

(a) We know that $0_{\mathcal{F}(\mathbb{R},\mathbb{R})} = 0$ is the zero mapping defined by

$$\forall x \in \mathbb{R} : 0(x) = 0$$

Therefore,

$$\forall x \in \mathbb{R} : 0(-x) = 0(x) = 0.$$

Thus, $0_{\mathcal{F}(\mathbb{R},\mathbb{R})} \in F_2$.

(b) Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in F_2$. Then, for all $x \in \mathbb{R}$, we have

$$\begin{cases} f(-x) = f(x) \\ g(-x) = g(x). \end{cases}$$

Let $x \in \mathbb{R}$. Then

$$(\alpha \cdot f + \beta \cdot g)(-x) = (\alpha \cdot f)(-x) + (\beta \cdot g)(-x) = \alpha \cdot f(-x) + \beta \cdot g(-x) = \alpha \cdot f(x) + \beta \cdot g(x) = (\alpha \cdot f + \beta \cdot g)(x)$$

This implies that

$$(\alpha \cdot f + \beta \cdot g) \in F_2$$

Thus, F_2 is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

The same method applies to show that F_3(the set of odd mappings) is also a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

For F_4 :

(a) We know that the zero polynomial $0_{\mathbb{R}[X]}$ of $\mathbb{R}[X]$, denoted by 0, is defined by

$$0 = 0 + 0.X + 0.X^2 + ... + ...$$

Moreover, it satisfies

$$0_{\mathbb{R}[X]}(0) = 0_{\mathbb{R}[X]}(1) = 0.$$

Therefore, $0_{\mathbb{R}[X]} \in F_4$..

(b) Let $\alpha, \beta \in \mathbb{R}$ and $P_1, P_2 \in F_4$. By definition, these polynomials satisfy

$$\begin{cases} P_1(0) = P_1(1) = 0 \\ P_2(0) = P_2(1) = 0 \end{cases}$$

Consider the linear combination $\alpha \cdot P_1 + \beta \cdot P_2$. We have

$$(\alpha \cdot P_1 + \beta \cdot P_2)(0) = \alpha \cdot P_1(0) + \beta \cdot P_2(0) = 0$$

and

$$(\alpha \cdot P_1 + \beta \cdot P_2)(1) = \alpha \cdot P_1(1) + \beta \cdot P_2(1) = 0$$

Thus $\alpha \cdot P_1 + \beta \cdot P_2 \in F_4$. which proves that F_4 is a subspace of $\mathbb{R}[X]$.

For F_5 **:** We recall that

$$\forall P, Q \in K[X], \forall \alpha \in K : \begin{cases} (a)\deg(0 = 0_{K[X]}) = -\infty.(\text{ by convention}) \\ (b)\deg(P+Q) \leq \max\{\deg(P), \deg(Q)\}. \\ (c)\deg(P.Q) = \deg(P) + \deg(Q). \\ (d)\text{If } \alpha \neq 0_K, \text{ then } \deg(\alpha P) = \deg(P). \end{cases} \text{, where } K \text{ is a field .}$$

(j) Since

$$\deg(0 = 0_{\mathbb{R}[X]}) = -\infty \leq n,$$

it follows that $0_{\mathbb{R}[X]} \in F_5$.

(jj) Let $P, Q \in F_5$. By definition of F_5, we have

$$\deg(P), \deg(Q) \leq n$$

According to property (b) of the degree of a sum of polynomials

$$\deg(P+Q) \leq \max\{\deg(P), \deg(Q)\} = n$$

Thus, $P + Q \in F_5$.

(jjj) Let $P \in F_5, \alpha \in \mathbb{R}$. Then

$$\deg(P) \leq n.$$

We distinguish two cases :

* If $\alpha = 0$, then $\alpha \cdot P = 0_{\mathbb{R}[X]} \in F_5$.

* If $\alpha \neq 0$, then according to property (d) of the degree of a product by a nonzero scalar

$$\deg(\alpha \cdot P) = \deg(P) \leq n$$

Thus, $\alpha \cdot P \in F_5$. This proves that F_5 is a subspace of $\mathbb{R}[X]$.

For F_6 :

We have

$$F_6 = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : f \text{ is bounded}\}$$
$$= \left\{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}), \exists M \in \mathbb{R}^+, \forall x \in \mathbb{R} : \left|f(x)\right| \leq M\right\}$$

(a) The zero mapping $0_{\mathcal{F}(\mathbb{R},\mathbb{R})} = 0$ is bounded, since for all $x \in \mathbb{R}$,

$$|0(x)| = |0| = 0 \leq M, \forall M \in \mathbb{R}^+$$

Therefore, $0_{\mathcal{F}(\mathbb{R},\mathbb{R})} = 0 \in F_6$.

(b) Stability under linear combinations : Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in F_6$. By definition of F_6, we have

$$\begin{cases} \exists M_1 \in \mathbb{R}^+, \forall x \in \mathbb{R} : \left|f(x)\right| \leq M_1 \\ \exists M_2 \in \mathbb{R}^+, \forall x \in \mathbb{R} : \left|g(x)\right| \leq M_2 \end{cases}$$

Then, for all $x \in \mathbb{R}$,

$$\left|(\alpha \cdot f + \beta \cdot g)(x)\right| \leq |\alpha| \left|f(x)\right| + \left|\beta\right| \left|g(x)\right|$$
$$\leq M_3 := |\alpha| M_1 + \left|\beta\right| M_2 \in \mathbb{R}^+$$

Thus, $\alpha \cdot f + \beta \cdot g \in F_6$. This proves that F_6 is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.