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Algebra II, Worksheet 2 answers

Exercise n°1 :

Solution :

1. The vector $x = (1, 1, 2)$ is a linear combination of $x_1 = (1, -1, 1)$ and $x_2 = (0, 1, a)$ if and only if there exist scalars $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$x = \alpha_1 x_1 + \alpha_2 x_2$$

This leads to the following system of equations

$$\begin{cases} \alpha_1 = 1 \\ -\alpha_1 + \alpha_2 = 1 \\ \alpha_1 + a\alpha_2 = 2 \end{cases}$$

Solving this system, we obtain the solution $\left[a = \frac{1}{2}, \alpha_1 = 1, \alpha_2 = 2\right]$.

2. Assume that the family of vectors $A = \{v_1, v_2, v_3, v_4\}$ is linearly independent. Then, the family $B = \{e_1, e_2, e_3, e_4\}$ is linearly independent if and only if the following condition holds

$$\forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in K : \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 = 0_E \Longrightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0_K.$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in K$, then

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 = 0_E \Longrightarrow (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) v_1 + (\alpha_2 + \alpha_3 + \alpha_4) v_2 + (\alpha_3 + \alpha_4) v_3 + \alpha_4 v_4 = 0_E$$

Since $A = \{v_1, v_2, v_3, v_4\}$ is linearly independent, the coefficients must satisfy

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0_K \\ \alpha_2 + \alpha_3 + \alpha_4 = 0_K \\ \alpha_3 + \alpha_4 = 0_K \\ \alpha_4 = 0_K \end{cases}$$

Solving this system, we deduce that

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0_K.$$

which proves that the family B is linearly independent.

3. We now show that $F_1 + F_2 = F$ is a vector subspace of E.

(a) **Non-empty :** Since F_1 and F_2 are subspaces, they both contain 0_E. Hence,

$$0_E = 0_E + 0_E \in F_1 + F_2.$$

Thus, $F_1 + F_2$ is non-empty set.

(b) Closed under addition : Let $x, y \in F$, so there exist $a_1, b_1 \in F_1$ and $a_2, b_2 \in F_2$ such that

$$\begin{cases} x = a_1 + a_2 \\ y = b_1 + b_2 \end{cases}$$

$$\Longrightarrow x + y = (a_1 + a_2) + (b_1 + b_2) =$$

$$= (a_1 + b_1) + (a_2 + b_2) \text{ (Since addition in } E \text{ is commutative and associative)}$$

Since F_1 and F_2 are vector subspaces, $a_3 = (a_1 + b_1) \in F_1$ and $b_3 = (a_2 + b_2) \in F_2$. Thus, $x + y = a_3 + b_3 \in F$, proving that F is closed under addition.

(c) Closed under scalar multiplication : Let $\alpha \in K$ and $x = x_1 + x_2 \in F$ with $x_1 \in F_1$ and $x_2 \in F_2$. Then,

$$\alpha x = \alpha (x_1 + x_2) = \alpha x_1 + \alpha x_2.$$

Since F_1 and F_2 are vector subspaces, $\alpha x_1 \in F_1$ and $\alpha x_2 \in F_2$. Thus, $\alpha x \in F$, proving that F is closed under scalar multiplication. Then F is a vector subspace of E.

Exercise n°2 : **Solution**

1. A set $B = \{f_1, f_2\}$ is linearly independent if and only if

$$\forall \alpha_1, \alpha_2 \in \mathbb{R} : \alpha_1 f_1 + \alpha_2 f_2 = 0_{\mathcal{F}(\mathbb{R},\mathbb{R})} \Longrightarrow \alpha_1 = \alpha_2 = 0.$$

where $0_{\mathcal{F}(\mathbb{R},\mathbb{R})}$ denotes the zero mapping.

Consider the equation $\alpha_1 f_1 + \alpha_2 f_2 = 0_{\mathcal{F}(\mathbb{R},\mathbb{R})}$ in the space $\mathcal{F}(\mathbb{R}, \mathbb{R})$. This equation is equivalent to

$$\forall x \in \mathbb{R} : \alpha_1 f_1(x) + \alpha_2 f_2(x) = 0$$

By choosing specific values : $x = 0$ and $x = \frac{\pi}{2}$, we obtain $\alpha_1 = 0$ and $\alpha_2 = 0$, respectively. Thus, the family B is linearly independent, and we conclude that B forms a basis for F.

2. (b) We know that f belongs to the vector subspace F spanned (or generated) by the family B, denoted $span(B)$ or $< B >$, if and only if there exist scalars $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$f = \lambda_1 f_1 + \lambda_2 f_2$$

$$\Longleftrightarrow \forall x \in \mathbb{R} : f(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x)$$

For any $x \in \mathbb{R}$, using trigonometric identities, we can express $f(x)$ as

$$f(x) = \cos(x + \alpha) = \cos x . \cos \alpha - \sin x . \sin \alpha$$

$$= \lambda_1 \cos x + \lambda_2 \sin x, \quad \text{where } \lambda_1 = \cos \alpha \text{ and } \lambda_2 = -\sin \alpha \text{ are constants in } \mathbb{R}$$

This shows that $f \in F$. The coordinates of f in the basis B of F are given by the pair $(\cos \alpha, -\sin \alpha)$.

Similarly, for the mapping g, we have

$$g(x) = \sin(x + \beta) = \sin x . \cos \beta + \cos x . \sin \beta$$

$$= \lambda_1 \cos x + \lambda_2 \sin x, \quad \text{where } \lambda_1 = \sin \beta \text{ and } \lambda_2 = \cos \beta \text{ are constants in } \mathbb{R}$$

This implies that $g \in F$. The coordinates of g in the basis B of F are given by the pair $(\sin \beta, \cos \beta)$.

Exercise n°3 :

Solution : To determine a basis of a vector subspace F, we first try to obtain a spanning family for F and then show that this family is linearly independent. Thus,

$$F = \left\{X = (x, y, z, t) \in \mathbb{R}^4 : y = -2x \text{ and } -x + 3z = t\right\} = \{X = (x, -2x, z, -x + 3z) : x, z \in \mathbb{R}\}$$

$$= \{X = x(1, -2, 0, -1) + z(0, 0, 1, 3) : x, z \in \mathbb{R}\} = \langle L_F = \{a = (1, -2, 0, -1), b = (0, 0, 1, 3)\}\rangle$$

$$= Span\,\{L_F = \{a = (1, -2, 0, -1), b = (0, 0, 1, 3)\}\}$$

Thus, F is a subspace of $\mathbb{R}^4$ spanned by the family of vectors $L_F = \{a = (1, -2, 0, -1), b = (0, 0, 1, 3)\}$. Moreover, it is easy to verify that L_F is linearly independent. Therefore, L_F forms a basis of F, and it follows that $\dim(F) = Card(L_F) = 2$.

Similarly, for G we find that G is a subspace of $\mathbb{R}^4$ spanned by the family of vectors

$$L_G = \{c = (-2, 1, 0, 0), d = (-3, 0, 1, 0), v = (-1, 0, 0, 1)\}$$

Moreover, it is easy to verify that L_G is linearly independent. Therefore, L_G forms a basis of G, which implies that $\dim(G) = Card(L_G) = 3$.

2. We have

$$\begin{aligned} F \cap G &= \left\{X = (x, y, z, t) \in \mathbb{R}^4 : X \in F \text{ and } X \in G\right\} = \\ &= \left\{X = (x, y, z, t) \in \mathbb{R}^4 : 2x + y = 0, -x + 3z - t = 0, x + 2y + 3z + t = 0\right\} \end{aligned}$$

$$= \left\{X = (x, y, z, t) \in \mathbb{R}^4 : x = \frac{3}{2}z, y = -3z, t = \frac{3}{2}z\right\}$$

(b) Basis of $F \cap G$: we have

$$F \cap G = \left\{X = \left(\frac{3}{2}z, -3z, z, \frac{3}{2}z\right) : z \in \mathbb{R}\right\} = span\left(L_{F\cap G} = \left\{m = \left(\frac{3}{2}, -3, 1, \frac{3}{2}\right)\right\}\right).$$

Thus, $F \cap G$ is a subspace of $\mathbb{R}^4$ generated by the family $L_{F\cap G} = \left\{m = \left(\frac{3}{2}, -3, 1, \frac{3}{2}\right)\right\}$. Since $m \neq 0_{\mathbb{R}^4}$, the set $F \cap G$ is linearly independent. Therefore, $L_{F\cap G}$ forms a basis of $F \cap G$ and we conclude that $\dim(F \cap G) = Card(L_{F\cap G}) = 1$.

3) According to Grassmann's formula, we have

$$\dim(F + G) = \dim(F) + \dim(G) - \dim(F \cap G) = 2 + 3 - 1 = 4.$$

Exercise n°4 : **Solution**

1. (a) Let $x \in \mathbb{R}$, then

$$\begin{cases} f_1(-x) = \frac{1}{2}\left[f(-x) + f(x)\right] = f_1(x) \\ f_2(-x) = \frac{1}{2}\left[f(-x) - f(x)\right] = -\frac{1}{2}\left[f(x) - f(-x)\right] = -f_2(x) \end{cases}$$

Thus, f_1 is even and f_2 is odd.

Now, let's prove that

$$EV(\mathbb{R}, \mathbb{R}) + OD(\mathbb{R}, \mathbb{R}) = \mathcal{F}(\mathbb{R}, \mathbb{R}).$$

j) Since $EV(\mathbb{R}, \mathbb{R}) + OD(\mathbb{R}, \mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$, we have

$$EV(\mathbb{R}, \mathbb{R}) + OD(\mathbb{R}, \mathbb{R}) \subset \mathcal{F}(\mathbb{R}, \mathbb{R}).$$

(jj) On the other hand, let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, then

$$f \in EV(\mathbb{R}, \mathbb{R}) + OD(\mathbb{R}, \mathbb{R}) \subset \mathcal{F}(\mathbb{R}, \mathbb{R}) \Longleftrightarrow \exists g_1 \in EV(\mathbb{R}, \mathbb{R}), \exists g_2 \in \Im(\mathbb{R}, \mathbb{R}) : f = g_1 + g_2$$

$$\Longleftrightarrow \exists g_1 \in EV(\mathbb{R}, \mathbb{R}), \exists g_2 \in \Im(\mathbb{R}, \mathbb{R}), \forall x \in \mathbb{R} : f(x) = g_1(x) + g_2(x).$$

For any $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and any $x \in \mathbb{R}$, we can write

$$f(x) = \frac{1}{2}(f(x)+f(x)) = \frac{1}{2}(f(x)+f(-x)-f(-x)+f(x)) = \frac{1}{2}(f(x)+f(-x))+\frac{1}{2}(f(x)-f(-x)) = \varphi_1(x)+\varphi_2(x)$$

where

$$\varphi_1(x) = \frac{1}{2}(f(x) + f(-x)) = f_1(x), \varphi_2(x) = \frac{1}{2}(f(x) - f(-x)) = f_2(x).$$

Since $f_1 \in EV(\mathbb{R}, \mathbb{R}), f_2 \in OD(\mathbb{R}, \mathbb{R})$, we conclude that

$$\exists g_1 = f_1 \in EV(\mathbb{R}, \mathbb{R}), \exists g_2 = f_2 \in OD(\mathbb{R}, \mathbb{R}) : f = g_1 + g_2$$

$$\Longrightarrow f \in EV(\mathbb{R},\mathbb{R}) + OD(\mathbb{R},\mathbb{R})$$

Thus

$$\mathcal{F}(\mathbb{R},\mathbb{R}) \subset EV(\mathbb{R},\mathbb{R}) + OD(\mathbb{R},\mathbb{R}).$$

Since we have both inclusions, we conclude

$$EV(\mathbb{R},\mathbb{R}) + OD(\mathbb{R},\mathbb{R}) = \mathcal{F}(\mathbb{R},\mathbb{R}).$$

(b) We know that the subspaces $EV(\mathbb{R},\mathbb{R})$ and $OD(\mathbb{R},\mathbb{R})$ are complementary in $\mathcal{F}(\mathbb{R},\mathbb{R})$ if and only if their direct sum equals $\mathcal{F}(\mathbb{R},\mathbb{R})$. That is,

$$\mathcal{F}(\mathbb{R},\mathbb{R}) = EV(\mathbb{R},\mathbb{R}) \oplus OD(\mathbb{R},\mathbb{R}) \iff$$

$$\iff \begin{cases} j) EV(\mathbb{R},\mathbb{R}) \cap OD(\mathbb{R},\mathbb{R}) = \{0_{\mathcal{F}(\mathbb{R},\mathbb{R})}\} \\ jj) \mathcal{F}(\mathbb{R},\mathbb{R}) = EV(\mathbb{R},\mathbb{R}) + OD(\mathbb{R},\mathbb{R}). \end{cases}$$

(j) Since $EV(\mathbb{R},\mathbb{R}) \cap OD(\mathbb{R},\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R},\mathbb{R})$, so the zero mapping $0_{\mathcal{F}(\mathbb{R},\mathbb{R})}$ belongs to both $EV(\mathbb{R},\mathbb{R})$ and $OD(\mathbb{R},\mathbb{R})$, it follows that

$$\{0_{\mathcal{F}(\mathbb{R},\mathbb{R})}\} \subset EV(\mathbb{R},\mathbb{R}) \cap OD(\mathbb{R},\mathbb{R}).$$

Now, let $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$, then

$$f \in EV(\mathbb{R},\mathbb{R}) \cap OD(\mathbb{R},\mathbb{R}) \Longrightarrow \begin{cases} f \in EV(\mathbb{R},\mathbb{R}) \\ \quad and \\ f \in OD(\mathbb{R},\mathbb{R}) \end{cases}$$

$$\Longrightarrow \begin{cases} f(x) = f(-x) \\ \quad and \\ f(x) = -f(-x) \end{cases}, \forall x \in \mathbb{R}.$$

$$\Longrightarrow 2f(x) = 0, \forall x \in \mathbb{R}$$

$$\Longrightarrow f(x) = 0, \forall x \in \mathbb{R}$$

$$\Longrightarrow f = 0_{\mathcal{F}(\mathbb{R},\mathbb{R})} \in \{0_{\mathcal{F}(\mathbb{R},\mathbb{R})}\}$$

$$\Longrightarrow EV(\mathbb{R},\mathbb{R}) \cap OD(\mathbb{R},\mathbb{R}) \subset \{0_{\mathcal{F}(\mathbb{R},\mathbb{R})}\}.$$

Since we have both inclusions, we conclude

$$EV(\mathbb{R},\mathbb{R}) \cap OD(\mathbb{R},\mathbb{R}) = \{0_{\mathcal{F}(\mathbb{R},\mathbb{R})}\}.$$

(jj) Since from part (a), we have $\mathcal{F}(\mathbb{R},\mathbb{R}) = EV(\mathbb{R},\mathbb{R}) + OD(\mathbb{R},\mathbb{R})$, it follows that

$$\mathcal{F}(\mathbb{R},\mathbb{R}) = EV(\mathbb{R},\mathbb{R}) \oplus OD(\mathbb{R},\mathbb{R})$$