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Algebra II, Worksheet 3 answers

Exercise No. 1 :
Solution :
1. Suppose that the mappings f and g are linear. Then, the composition

$$E \xrightarrow{f} F \xrightarrow{g} G$$

is defined, and thus

$$g \circ f : E \longrightarrow G$$

We know that

$$g \circ f \text{ is linear (or a vector space morphism)} \iff$$
$$\iff \begin{cases} (a_1)\forall x, y \in E : (g \circ f)(x+y) = (g \circ f)(x) + (g \circ f)(y) \\ (a_2)\forall \lambda \in K : \forall x \in E : (g \circ f)(\lambda \cdot x) = \lambda \cdot (g \circ f)(x) \end{cases}$$
$$\iff \forall \alpha, \beta \in K : \forall x, y \in E : (g \circ f)(\alpha \cdot x + \beta \cdot y) = \alpha \cdot (g \circ f)(x) + \beta \cdot (g \circ f)(y).$$

Let $\alpha, \beta \in K$ and $x, y \in E$ .Then

$$(g \circ f)(\alpha \cdot x + \beta \cdot y) = g(f(\alpha \cdot x + \beta \cdot y)) = g(\alpha \cdot f(x) + \beta \cdot f(y)) \text{ (since } f \text{ is linear)}$$
$$= \alpha \cdot g(f(x)) + \beta \cdot g(f(y)) \text{ (since } g \text{ is linear)}$$
$$= \alpha \cdot (g \circ f)(x) + \beta \cdot (g \circ f)(y).$$

Therefore, $g \circ f$ is linear, and hence $g \circ f \in \mathcal{L}_{\mathbb{K}}(E, G)$, where $\mathcal{L}_{\mathbb{K}}(E, G)$ denotes the set of all linear mappings from E to G over the field K.

2. (a) Since f is defined from $\mathbb{R}^3$ to $\mathbb{R}^3$, it suffices to determine $f(x)$ for any $x \in \mathbb{R}^3$. Let $x \in \mathbb{R}^3$, since B is a basis of $\mathbb{R}^3$, there exist unique scalars $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 .$$

Thus, we have

$$f(x) = f(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) = \lambda_1 f(e_1) + \lambda_2 f(e_2) + \lambda_3 f(e_3) \text{ (since } f \text{ is linear)}$$
$$= \frac{\lambda_1}{3}(-e_1 + 2e_2 + 2e_3) + \frac{\lambda_2}{3}(2e_1 - e_2 + 2e_3) + \frac{\lambda_3}{3}(2e_1 + 2e_2 - e_3)$$
$$= \left(\frac{-\lambda_1}{3} + \frac{2\lambda_2}{3} + \frac{2\lambda_3}{3}\right)e_1 + \left(\frac{2\lambda_1}{3} - \frac{\lambda_2}{3} + \frac{2\lambda_3}{3}\right)e_2 + \left(\frac{2\lambda_1}{3} + \frac{2\lambda_2}{3} - \frac{\lambda_3}{3}\right)e_3$$
$$= \frac{1}{3}(-\lambda_1 + 2\lambda_2 + 2\lambda_3)e_1 + \frac{1}{3}(2\lambda_1 - \lambda_2 + 2\lambda_3)e_2 + \frac{1}{3}(2\lambda_1 + 2\lambda_2 - \lambda_3)e_3$$

Thus, the linear mapping f can be expressed as

$$\begin{array}{llll} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & x = (\lambda_1, \lambda_2, \lambda_3) & \longmapsto & f(x) = \left(\frac{1}{3}(-\lambda_1 + 2\lambda_2 + 2\lambda_3), \frac{1}{3}(2\lambda_1 - \lambda_2 + 2\lambda_3), \frac{1}{3}(2\lambda_1 + 2\lambda_2 - \lambda_3)\right). \end{array}$$

(j)We have

$$f(v_1) = f(e_1 - e_2) = f(e_1) - f(e_2) = e_2 - e_1 = -(e_1 - e_2) = -v_1,$$

so $v_1 \in F_1$. The same method applies to v_2 and v_3.

(k) Recall that the family B' forms a basis of $\mathbb{R}^3$ if and only if it is linearly independent and spans $\mathbb{R}^3$ ($span(B') = \mathbb{R}^3$). Since the dimension of $\mathbb{R}^3$ is equal to the cardinality of B', it suffices to show that B' is linearly independent.

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0_{\mathbb{R}^3}.$$

This leads to $\lambda_1 = \lambda_2 = \lambda_3 = 0$, hence B' is linearly independent and therefore B' forms a new basis for $\mathbb{R}^3$.

(l) To determine the mapping $f^2 = f \circ f$, it suffices to compute $f^2(x)$ for all $x \in \mathbb{R}^3$, since f^2 is defined from $\mathbb{R}^3$ to $\mathbb{R}^3$.

Let $x \in \mathbb{R}^3$. Since B' is a basis of $\mathbb{R}^3$, we can express x in the form

$$x = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

then we have (Applying f twice, we obtain)

$$f^2(x) = (f \circ f)(x) = f(f(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3)) = f(\lambda_1 f(v_1) + \lambda_2 f(v_2) + \lambda_3 f(v_3))$$

$$= f(-\lambda_1 v_1 - \lambda_2 v_2 + \lambda_3 v_3) = -\lambda_1 f(v_1) - \lambda_2 f(v_2) + \lambda_3 f(v_3)$$

$$= \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = x.$$

Thus, we conclude that

$$f^2 = f \circ f = Id_{\mathbb{R}^3},$$

is the identity mapping on $\mathbb{R}^3$.

We know that the mapping f is bijective if and only if there exists a mapping $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

$$g \circ f = Id_{\mathbb{R}^3} \text{ et } f \circ g = Id_{\mathbb{R}^3}$$

with $f^{-1} = g$. Since

$$f^2 = f \circ f = Id_{\mathbb{R}^3}$$

it follows that f is bijective, and the inverse of f is f itself (i.e., $f^{-1} = f$).

Exercise No. 2 :

Solution :

1. (i) We know that f is an endomorphism of the vector space $\mathbb{R}_2[X]$ if and only if :
(a) f is a morphism of the vector space $\mathbb{R}_2[X]$ (i.e., a linear mapping).
(b) f is defined from $\mathbb{R}_2[X]$ to $\mathbb{R}_2[X]$, that is $f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$.
(a) Let t $\lambda_1, \lambda_2 \in \mathbb{R}$ and $P_1, P_2 \in \mathbb{R}_2[X]$. Then,

$$f(\lambda_1 P_1 + \lambda_2 P_2) = -\frac{(X+1)^2}{2}(\lambda_1 P_1 + \lambda_2 P_2)^{(2)} + (X+1)(\lambda_1 P_1 + \lambda_2 P_2)^{(1)} = \lambda_1 f(P_1) + \lambda_2 f(P_2).$$

This shows that f is linear.
(b) On the other hand, let $P = a_0 + a_1 X + a_2 X^2 \in \mathbb{R}_2[X]$. Then,

$$P' = a_1 + 2a_2 X, P'' = 2a_2$$

We obtain

$$f(P) = a_1 - a_2 + a_1 X + a_2 X^2.$$

Since $a_2 \in \mathbb{R}$, it follows that $\deg(f(p)) \leq 2$. Therefore, $f(P) \in \mathbb{R}_2[X]$, which shows that f is an endomorphism of $\mathbb{R}_2[X]$.

(j) We have

$$f \circ f = f \Longleftrightarrow \forall P \in \mathbb{R}_2[X] : (f \circ f)(P) = f(P)$$

since $f \circ f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$ and $f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$

Let $P = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]$. Then,

$$f(P) = a_1 - a_2 + a_1X + a_2X^2 = c_0 + c_1X + c_2X^2$$

where

$$c_0 = a_1 - a_2, c_1 = a_1, c_2 = a_2.$$

Thus, we obtain

$$(f \circ f)(P) = f(f(P)) = f(c_0 + c_1X + c_2X^2) = c_1 - c_2 + c_1X + c_2X^2 = a_1 - a_2 + a_1X + a_2X^2 = f(P).$$

This proves that $f \circ f = f$.

2. (i) We know that the kernel of f, denoted $\ker(f)$, is defined by

$$\ker(f) = \left\{P = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X] : f(P) = 0_{\mathbb{R}_2[X]}\right\}$$

$$= \left\{P = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X] : a_1 - a_2 + a_1X + a_2X^2 = 0 + 0X + 0X^2\right\}$$

$$= \left\{P = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X] : a_1 = a_2 = 0\right\}$$

$$= \{P = a_0 = a_0 \cdot 1 : a_0 \in \mathbb{R}\} \text{ (the kernel consists of constant polynomials)}$$

$$= span(L_1 = \{R_1 = 1\})$$

is the vector subspace of $\mathbb{R}_2[X]$ spanned by the family L_1. Since, $R_1 = 1 \neq 0_{\mathbb{R}_2[X]}$, so L_1 is linearly independent, and thus L_1 forms a basis of $\ker(f)$.

(ii) The image of f, denoted by $Im(f)$ or $f(\mathbb{R}_2[X])$, is the subset defined as

$$Im(f) = f(E) = \left\{f(P) : P = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]\right\}$$

$$= \left\{a_1 - a_2 + a_1X + a_2X^2 : a_1, a_2 \in \mathbb{R}_2[X]\right\}$$

$$= \left\{a_1(X + 1) + a_2(X^2 - 1) : a_1, a_2 \in \mathbb{R}_2[X]\right\}$$

$$= \{a_1Q_1 + a_2Q_2 : a_1, a_2 \in \mathbb{R}_2[X]\}$$

$$= span(L_2 = \{Q_1, Q_2\})$$

is the vector subspace of $\mathbb{R}_2[X]$ spanned by the family L_2.

Moreover, it is easy to verify that the family L_2 is linearly independent. Hence, L_2 is a basis for $Im(f)$. We then obtain that the rank of f is :

$$rank(f) = dim2061(Im(f)) = 2.$$

Other method : By the Rank Theorem, we have

$$\dim \mathbb{R}_2[X] = \dim(\ker(f)) + \dim(Im(f)) = \dim(\ker(f)) + rank(f).$$

which implies

$$rank(f) = \dim \mathbb{R}_2[X] - \dim(\ker(f)) = 3 - 1 = 2$$

Exercise No. 3 :

Solution

1. Suppose $f(L)$ is linearly independent in F. Consider scalars $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$ such that

$$\lambda_1x_1 + \lambda_2x_2 + ... + \lambda_nx_n = 0_E.$$

Applying f and using its linearity, we obtain

$$f(\lambda_1x_1 + \lambda_2x_2 + ... + \lambda_nx_n) = \lambda_1f(x_1) + \lambda_2f(x_2) + ... + \lambda_nf(x_n) = f(0_E) = 0_F.$$

Since $f(L)$ is linearly independent, it follows that all the scalars $\lambda_1, \lambda_2, ..., \lambda_n$ are zero. Thus, L is also linearly independent in E.
2. Suppose L is linearly independent in E and f is injective. Consider scalars $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$ such that

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + ... + \lambda_n f(x_n) = 0_F,$$

then, by linearity of f,

$$f(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n) = f(0_E).$$

Since f is injective,

$$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n = 0_E.$$

As L is linearly independent, all $\lambda_1, \lambda_2, ..., \lambda_n$ must be zero. Thus, $f(L)$ is linearly independent in F.

3. If L is linearly dependent in E, there exist scalars $\lambda_1, \lambda_2, ..., \lambda_n$, not all zero, such that

$$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n = 0_E.$$

Applying to both sides and using linearity, we obtain

$$f(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n) = \lambda_1 f(x_1) + \lambda_2 f(x_2) + ... + \lambda_n f(x_n) = f(0_E) = 0_F$$

Since at least one λ_i is nonzero, this implies that $f(L)$ is linearly dependent in F.