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First year of Computer Science License 2024/2025 Algebra II, Worksheet 4

Exercise No. 1 : Consider the following matrix$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$.

1. Compute the powers A^2, A^3. Deduce the expression for A^n for all integers $n \geq 3$.
2. For every real number $x \in \mathbb{R}$, define the matrix $M(x) = I_3 + xA + \frac{x^2}{2}A^2$.
(a) Express the matrix $M(x)$ explicitly.
(b) Prove that $\forall x, y \in \mathbb{R} : M(x) \cdot M(y) = M(x+y)$. Determine a real number $x' \in \mathbb{R}$ such that $M(x) \cdot M(x') = I_3$. Deduce that $M(x)$ is invertible and determine $M(x)^{-1}$.

Exercise No. 2 : Let $E = \mathcal{M}_n(\mathbb{R})$ be the vector space of square matrices of order n with real entries. Consider two subsets F_1 and F_2 of E defined by

$$F_1 = \left\{A \in \mathcal{M}_n(\mathbb{R}) :^t A = A\right\}, F_2 = \left\{A \in \mathcal{M}_n(\mathbb{R}) :^t A = -A\right\}$$

where ${}^tA$ denotes the transpose of matrix A.
1. Prove that F_1 and F_2 are vector subspaces of $\mathcal{M}_n(\mathbb{R})$.
2. For any matrix$A \in \mathcal{M}_n(\mathbb{R})$, show that $B_1 = \frac{1}{2}({}^tA + A) \in F_1$ and $B_2 = \frac{1}{2}(A -^t A) \in F_2$.
3. Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$, decompose A as the sum of a matrix in F_1 and a matrix in F_2.

Exercise No. 3 : Consider the linear mapping f defined by

$$f : \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x,y,z) & \longmapsto & f(x,y,z) = (3x - y + z, -x - 2y - 5z, x + y + 3z) \end{array}$$

Let $B = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ be the canonical (or standard) basis of $\mathbb{R}^3$.
1. Find the matrix $A = \underset{B}{Mat}(f)$ associated with f with respect to the basis B.
2. Let $B' = \left\{e_1' = (-4,1,3), e_2' = (2,0,-1), e_3' = (-1,1,1)\right\}$ be a new basis of $\mathbb{R}^3$.
(a) Determine the change-of-basis matrix P from B to B' and compute its inverse P^{-1}.
(b) For the vector $v = (1,2,-1) \in \mathbb{R}^3$, find the coordinates of v in the new basis B'.
(c) Compute the matrix $A' = \underset{B'}{Mat}(f)$ associated with f with respect to the basis B'.

Exercise No. 4 : (Supplementary Exercise) Let $E = M_2(\mathbb{R})$ be the vector space over $\mathbb{R}$ of square matrices of order 2 with real entries defined as $E = M_2(\mathbb{R}) = \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R}\right\}$
1. Find a basis B of $M_2(\mathbb{R})$.
2. Consider a matrix $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in M_2(\mathbb{R})$ and a linear mapping $f : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ defined by

$$\forall A \in M_2(\mathbb{R}) : f(A) = M \cdot A.$$

a) Prove that f is linear and find $D = \underset{B}{Mat}(f)$, the matrix associated with f relative to the basis B.

Exercise No. 5 : (Supplementary Exercise) Consider the linear mapping $f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$ defined by

$$\forall P \in \mathbb{R}_2[X] : f(P) = P^{(1)} - (X+1)P^{(2)}$$

where $P^{(1)}$ and $P^{(2)}$ denote the first and second derivatives of P, respectively.
1. Prove that f is a linear mapping. Determine the kernel $\ker(f)$, the image $Im(f)$, and their respective dimensions. Is f injective ? Surjective ?.
2. Find the matrix $A = \underset{B}{Mat}(f)$ associated with f with respect to the canonical basis B.
3. Let $B' = \{Q_0 = X - 1, Q_1 = X^2 - 1, Q_2 = X^2 + 1\}$ be a new basis of $\mathbb{R}_2[X]$.
(a) Compute the change-of-basis matrix H from B to B', and find H^{-1}. Determine the coordinates of the polynomial $P = 1 + X + X^2$ in the basis B'.